

# Cyclic polynomials in Dirichlet-type Spaces of the unit bidisk

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# Preliminaries

- $\mathbb{C}$  denotes the complex plane.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  is the open unit disk.
- $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$  is the unit circle.
- $\mathcal{B}(\mathcal{H})$  denotes the set of all bounded linear operators on a Hilbert space  $\mathcal{H}$ .
- Let  $M$  be a closed subspace of  $\mathcal{H}$  and let  $T \in \mathcal{B}(\mathcal{H})$ . We say  $M$  is  **$T$ -invariant** if

$$T(M) \subseteq M.$$

- Given a function  $f$ , the set  $\mathcal{Z}(f)$  denotes the zero set of  $f$ .

# Cyclic Vector for a Single Operator

Let  $T \in \mathcal{B}(\mathcal{H})$ . A vector  $x \in \mathcal{H}$  is called **cyclic for**  $T$  if

$$[x]_T := \overline{\text{span}\{T^n x : n = 0, 1, 2, \dots\}} = \mathcal{H}.$$

Equivalently,

$$\overline{\{p(T)x : p \text{ is a polynomial}\}} = \mathcal{H}.$$

In other words, the smallest closed  $T$ -invariant subspace containing  $x$  is  $\mathcal{H}$ .

# Hardy Space $H^2(\mathbb{D})$

- The Hardy space  $H^2(\mathbb{D})$  consists of all analytic functions in the unit disk

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

such that

$$\|f\|_{H^2}^2 := \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

- The shift operator  $S : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  is defined by

$$(Sf)(z) = zf(z).$$

## Remark

- Polynomials are dense in  $H^2(\mathbb{D})$ .
- Monomials are orthogonal to each other in  $H^2(\mathbb{D})$ .

# Cyclic Functions and Outer Functions

- A function  $f \in H^2(\mathbb{D})$  is called *cyclic* if

$$[f] := \overline{\text{span}}\{z^n f : n \geq 0\} = H^2(\mathbb{D}).$$

- A consequence of Beurling's theorem yields that

$$f \text{ is cyclic in } H^2(\mathbb{D}) \iff f \text{ is outer,}$$

meaning

$$\log |f(0)| = \int_0^{2\pi} \log |f^*(e^{i\theta})| \frac{d\theta}{2\pi} \quad \text{and} \quad f(0) \neq 0.$$

Here  $f^*(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$  which is known to exist for a.e.  $\theta \in \mathbb{T}$  and for every  $f \in H^2(\mathbb{D})$ .

# Dirichlet-Type Spaces in the Unit Disk

Let  $\alpha \in \mathbb{R}$ . The **Dirichlet-type space**  $D_\alpha$  in the unit disk  $\mathbb{D}$  consists of all holomorphic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

such that

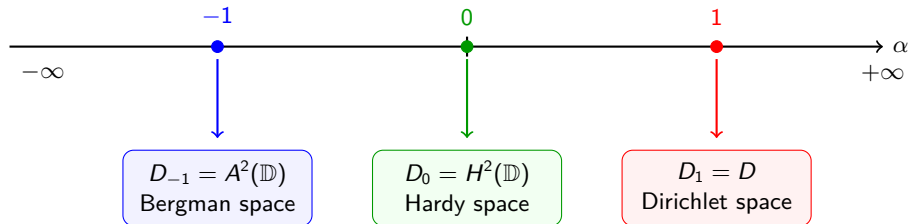
$$\|f\|_{D_\alpha}^2 := \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty.$$

**Remark.**

- $D_\alpha$  is a reproducing kernel Hilbert space with the inner product

$$\langle f, g \rangle_{D_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha a_n \overline{b_n}, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

# The Scale of Dirichlet-Type Spaces $D_\alpha$



- Smaller values of  $\alpha$  correspond to larger spaces.
- Increasing  $\alpha$  imposes stronger decay on the Taylor coefficients.

## Cyclic Functions in $D_\alpha$

$$D_\alpha = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty \right\}$$

- $\alpha = 0$  :  $f$  cyclic  $\iff f$  outer.
- $\alpha = 1$  : Brown–Shields conjecture (1984) :

$$f \text{ cyclic} \iff \begin{cases} f \text{ outer,} \\ c(\mathcal{Z}(f) \cap \mathbb{T}) = 0. \end{cases}$$

Here  $c(\cdot)$  denote the logarithmic capacity.

- $\alpha > 1$  :  $D_\alpha$  is an algebra.

$$f \text{ cyclic} \iff \mathcal{Z}(f) \cap \overline{\mathbb{D}} = \emptyset.$$

# Important Facts

- Since polynomials are dense in  $D_\alpha$ , the constant function 1 is cyclic in  $D_\alpha$ .
- If  $g \in [f]$ , then

$$[g] \subseteq [f].$$

Consequently, to prove that  $f$  is cyclic in  $D_\alpha$ , it suffices to construct a sequence of polynomials

$$p_n \in \mathbb{C}[z]$$

such that

$$\|p_n f - 1\|_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- Suppose

$$f = gh,$$

where  $g, h \in \mathbb{C}[z]$  are nonconstant polynomials.

Then

$$f \text{ is cyclic in } D_\alpha \iff g \text{ and } h \text{ are cyclic in } D_\alpha.$$

## Some more results

**A Necessary condition :** Let  $f \in D_\alpha(\mathbb{D})$  be cyclic. Then  $f$  has no zeros in  $\mathbb{D}$ .

**A Sufficient condition :** Let  $f \in D_\alpha(\mathbb{D})$  be such that

$$\mathcal{Z}(f) \cap \bar{\mathbb{D}} = \emptyset.$$

Then  $f$  is cyclic in  $D_\alpha$ .

**Theorem (Brown & Shields)**

Let  $\zeta \in \mathbb{T}$ . The polynomial  $z - \zeta$  is cyclic in  $D_\alpha$  if and only if  $\alpha \leq 1$ .

# Cyclic Vector for a Commuting Pair of Operators

Let  $T_1, T_2 \in \mathcal{B}(\mathcal{H})$  be commuting operators :

$$T_1 T_2 = T_2 T_1.$$

A vector  $x \in \mathcal{H}$  is called **cyclic for**  $(T_1, T_2)$  if

$$\overline{\text{span}}\{T_1^{\alpha_1} T_2^{\alpha_2} x : (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2\} = \mathcal{H}.$$

Equivalently,

$$\overline{\{p(T_1, T_2)x : p \in \mathbb{C}[z_1, z_2]\}} = \mathcal{H}.$$

That is, the smallest closed subspace invariant under both  $T_1$  and  $T_2$  containing  $x$  is  $\mathcal{H}$ .

# Dirichlet-type spaces

For  $\alpha \in \mathbb{R}$ , consider

- the Dirichlet-type space denoted by  $\mathfrak{D}_\alpha$  on  $\mathbb{D}^2$ , consisting of all holomorphic functions  $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$  such that

$$\|f\|_{\mathfrak{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{kl}|^2 < \infty.$$

- the Dirichlet-type space denoted by  $\mathcal{D}_\alpha$  on  $\mathbb{D}^2$ , consisting of all holomorphic functions  $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$  such that

$$\|f\|_{\mathcal{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

## Cyclic polynomials in $\mathcal{D}_\alpha$

Theorem (Bénéteau–Khavinson–Kosiński–Liaw–Seco–Sola, *Trans. Amer. Math. Soc.*)

Let  $p \in \mathbb{C}[z_1, z_2]$  be an irreducible polynomial with no zeros in the bidisk. We have the following :

- 1 If  $\alpha \leq \frac{1}{2}$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$ .
- 2 If  $\frac{1}{2} < \alpha \leq 1$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty or finite, **or**  $p$  is a constant multiple of  $\zeta - z_1$  or of  $\zeta - z_2$  for some  $\zeta \in \mathbb{T}$ .
- 3 If  $\alpha > 1$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty.

**Question** : Characterize cyclic polynomials in Dirichlet type space  $\mathcal{D}_\alpha$ ?

Some partial answers are obtained by Torkinejad Ziarati and the following problem was asked in his recent paper :

**Problem** : Is the polynomial  $2 - z_1 - z_2$  cyclic in  $\mathcal{D}_\alpha$  for  $\frac{3}{2} \leq \alpha \leq 2$ ?

## Some comments

Recall that the boundary of  $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$  is

$$\partial(\mathbb{D}^2) = (\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{T}).$$

Thus, zeros of a cyclic polynomial may occur on :

$$\mathbb{T}^2, \quad \mathbb{D} \times \mathbb{T}, \quad \mathbb{T} \times \mathbb{D}.$$

But if  $p \in \mathbb{C}[z_1, z_2]$  is an irreducible polynomial with

- no zeros in  $\mathbb{D}^2$ ,
- depends on both variables.

Then  $p$  cannot vanish on  $\mathbb{D} \times \mathbb{T}$  and  $\mathbb{T} \times \mathbb{D}$ . This is an application of Hurwitz's theorem.

# Cyclic polynomials in $\mathcal{D}_\alpha$

## Theorem

Let  $p \in \mathbb{C}[z_1, z_2]$  be an irreducible polynomial with no zeros in the bidisk. We have the following :

- 1 If  $\alpha \leq 1$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$ .
- 2 If  $1 < \alpha \leq 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty or finite.
- 3 If  $\alpha > 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty.

# Model Polynomials

To understand cyclicity in  $\mathcal{D}_\alpha$ , it is natural to first investigate a few model polynomials.

The most important examples are :

$$1 - z_1, \quad 1 - z_1 z_2, \quad 2 - z_1 - z_2.$$

These polynomials represent the fundamental types of boundary zero sets on  $\mathbb{T}^2$ .

- $1 - z_1$  :

$$\mathcal{Z}(p) \cap \partial(\mathbb{D}^2) = \{(1, z) : z \in \overline{\mathbb{D}}\}.$$

- $1 - z_1 z_2$  :

$$\mathcal{Z}(p) \cap \mathbb{T}^2 = \{(e^{it}, e^{-it}) : t \in \mathbb{R}\},$$

a distinguished curve on  $\mathbb{T}^2$ .

- $2 - z_1 - z_2$  :

$$\mathcal{Z}(p) \cap \mathbb{T}^2 = \{(1, 1)\},$$

an isolated boundary zero.

Thus these examples capture the main geometric possibilities for boundary zeros of polynomials on the bidisk.

# A one variable polynomial

Let

$$p(z) = z - a, \quad |a| = 1,$$

and define

$$P(z_1, z_2) := p(z_1).$$

Then  $P$  is cyclic in  $\mathcal{D}_\alpha$  if and only if

$$\alpha \leq 1.$$

**Idea of the proof.** For  $f \in D_\alpha$ , define

$$F(z_1, z_2) := f(z_1).$$

Then

$$\|f\|_{D_\alpha} = \|F\|_{\mathcal{D}_\alpha}.$$

Thus cyclicity of  $P(z_1, z_2) = p(z_1)$  in  $\mathcal{D}_\alpha$  reduces to cyclicity of  $p$  in the one-variable space  $D_\alpha$ .

Using the Brown–Shields theorem :

$$p \text{ is cyclic in } D_\alpha \iff \alpha \leq 1.$$

# Cyclicity of the polynomial $1 - z_1 z_2$

The polynomial

$$1 - z_1 z_2$$

is cyclic in  $\mathcal{D}_\alpha$  if and only if

$$\alpha \leq 1.$$

**Key observation.**

If

$$F(z_1, z_2) = f(z_1 z_2), \quad \text{for some one variable function } f,$$

then for  $\alpha > 0$ ,

$$\|f\|_{D_\alpha} \leq \|F\|_{\mathcal{D}_\alpha} \leq 2^\alpha \|f\|_{D_\alpha}.$$

Hence the map

$$f(z) \mapsto f(z_1 z_2)$$

preserves cyclicity up to norm equivalence.

# Main Theorem

## Theorem

Let  $\alpha \leq 2$ . Then

$$p(z_1, z_2) = 2 - z_1 - z_2$$

is cyclic in  $\mathcal{D}_\alpha$ .

Equivalently, if a sequence of complex numbers  $\{b_{k,l}\}_{k,l \in \mathbb{Z}_+}$  satisfies

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \in \mathbb{Z}_+,$$

and

$$\sum_{k,l \in \mathbb{Z}_+} \frac{|b_{k,l}|^2}{(k+l+1)^2} < \infty,$$

then  $b_{k,l} = 0$  for all  $k, l \in \mathbb{Z}_+$ .

**Proof** : Assume

$$f \in \mathcal{D}_2, \quad f \perp [p] \quad (\text{to prove : } f = 0).$$

Write

$$f(z_1, z_2) = \sum_{i,j=0}^{\infty} \frac{b_{ij}}{(i+j+1)^2} z_1^i z_2^j. \quad \left( \|f\|_{\mathcal{D}_2}^2 = \sum_{i,j=0}^{\infty} \frac{|b_{ij}|^2}{(i+j+1)^2} \right)$$

Since

$$f \perp z_1^k z_2^l (2 - z_1 - z_2),$$

we obtain

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \geq 0.$$

Define

$$g(z_1, z_2) := \sum_{i, j \geq 0} b_{ij} z_1^i z_2^j.$$

Since  $f \in \mathcal{D}_2$ , we have

$$g \in \mathcal{D}_{-2}.$$

Using the recurrence relation,

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1},$$

we obtain

$$(z_1 + z_2 - 2z_1z_2)g(z_1, z_2) = z_1g(z_1, 0) + z_2g(0, z_2).$$

We now introduce the substitutions

$$z_1 = \frac{\zeta}{\zeta - 1}, \quad z_2 = \frac{\zeta}{\zeta + 1}.$$

These satisfy :

$$z_1 \in \mathbb{D} \iff \Re \zeta < \frac{1}{2},$$

and

$$z_2 \in \mathbb{D} \iff \Re \zeta > -\frac{1}{2}.$$

# Construction of an Entire Function

Substituting into the functional equation gives

$$\frac{\zeta}{\zeta-1}g\left(\frac{\zeta}{\zeta-1}, 0\right) + \frac{\zeta}{\zeta+1}g\left(0, \frac{\zeta}{\zeta+1}\right) = 0.$$

Define

$$h(\zeta) = \begin{cases} \frac{1}{\zeta-1}g\left(\frac{\zeta}{\zeta-1}, 0\right), & \Re\zeta < \frac{1}{2}, \\ -\frac{1}{\zeta+1}g\left(0, \frac{\zeta}{\zeta+1}\right), & \Re\zeta > -\frac{1}{2}. \end{cases}$$

Hence  $h$  is a well-defined entire function.

Using the fact that

$$g \in \mathcal{D}_{-2},$$

one obtains

$$\int_{|\zeta|>1} \frac{|h(\zeta)|^2}{|\zeta|^4} dA(\zeta) < \infty.$$

This implies that  $h$  is a polynomial of degree at most 1 :

$$h(\zeta) = a(\zeta - 1) + b.$$

A further integrability argument shows that

$$a = 0.$$

Hence

$$h(\zeta) = b.$$

Using the definition of  $h$ ,

$$g(z_1, 0) = \frac{b}{z_1 - 1}, \quad g(0, z_2) = \frac{b}{z_2 - 1}.$$

Substituting into the functional equation yields

$$(z_1 + z_2 - 2z_1z_2)g(z_1, z_2) = \frac{bz_1}{z_1 - 1} + \frac{bz_2}{z_2 - 1}.$$

Solving,

$$g(z_1, z_2) = -\frac{b}{(1 - z_1)(1 - z_2)}.$$

Using the geometric series expansion,

$$g(z_1, z_2) = -b \sum_{k, l \geq 0} z_1^k z_2^l.$$

Hence

$$a_{kl} = -b, \quad k, l \geq 0.$$

## Conclusion

Now compute

$$\|g\|_{-2}^2 = |b|^2 \sum_{k,l \geq 0} \frac{1}{(k+l+1)^2}.$$

Grouping terms according to  $n = k + l$ ,

$$\sum_{k,l \geq 0} \frac{1}{(k+l+1)^2} = \sum_{n=0}^{\infty} \frac{n+1}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

This series diverges.

Therefore,

$$g \notin \mathcal{D}_{-2},$$

unless

$$b = 0.$$

Hence

$$g = 0, \quad f = 0.$$

Thus

$$[p]^\perp = \{0\},$$

and therefore

$$p(z_1, z_2) = 2 - z_1 - z_2$$

is cyclic in  $\mathcal{D}_\alpha$  for all  $\alpha \leq 2$ .

## Lemma

Let  $f \in \mathbb{C}[z_1, z_2]$ . Assume that

- $f$  has no zeros in  $\mathbb{D}^2$
- $f$  has finitely many zeros on  $\mathbb{T}^2$ , i.e.,

$$\mathcal{Z}(f) \cap \mathbb{T}^2 = \{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}.$$

Then for any integer  $m$  there exists sufficiently large  $N$  such that the function

$$Q(z_1, z_2) = \frac{\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)^N}{f(z_1, z_2)}$$

is  $m$ -times differentiable on  $\mathbb{T}^2$ .

# Polynomials with Finitely Many Boundary Zeros

## Theorem

Let  $p \in \mathbb{C}[z_1, z_2]$  be a polynomial having

- no zeros in  $\mathbb{D}^2$
- only finitely many zeros on  $\mathbb{T}^2$ .

Then  $p$  is cyclic in  $\mathcal{D}_\alpha$  for every

$$\alpha \leq 2.$$

## Proof.

There exists a function

$$Q(z_1, z_2) = \frac{\left(\prod_{i=1}^k (2 - \zeta_i^{-1}z_1 - \eta_i^{-1}z_2)\right)^N}{p(z_1, z_2)} = \frac{g(z_1, z_2)}{p(z_1, z_2)},$$

where  $(\zeta_i, \eta_i) \in \mathbb{T}^2$ , such that  $Q$  is twice continuously differentiable on  $\mathbb{T}^2$ .

Hence its Fourier coefficients satisfy

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k+1)^2 (l+1)^2 < \infty.$$

Since

$$(k + l + 1)^2 \leq (k + 1)^2(l + 1)^2,$$

we obtain

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k + l + 1)^2 < \infty.$$

Therefore,

$$Q \in \mathcal{D}_\alpha, \quad \alpha \leq 2.$$

Hence

$$g(z_1, z_2) \in p\mathcal{D}_\alpha.$$

On the other hand,

$$g(z_1, z_2) = \left( \prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2) \right)^N$$

is cyclic in  $\mathcal{D}_2$ .

Since  $p$  is a multiplier of  $\mathcal{D}_2$ , it follows that  $p$  is cyclic in  $\mathcal{D}_2$ , and consequently in  $\mathcal{D}_\alpha$  for all  $\alpha \leq 2$ .

## Question

For  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ , consider the Dirichlet-type space denoted by  $\mathcal{D}_{\vec{\alpha}}$  on  $\mathbb{D}^2$ , consisting of all holomorphic functions

$$f(z, w) = \sum_{k, l=0}^{\infty} a_{kl} z^k w^l$$

such that

$$\|f\|_{\mathcal{D}_{\vec{\alpha}}}^2 := \sum_{k, l=0}^{\infty} (k+1)^{\alpha_1} (l+1)^{\alpha_2} (k+l+1)^{\alpha_3} |a_{kl}|^2 < \infty.$$

**Question :** Characterize cyclic polynomial in  $\mathcal{D}_{\vec{\alpha}}$ .

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*Thank You For Your Attention !*