INVARIANT SUBSPACES AND THE C_{00}-PROPERTY OF 3-BROWNIAN SHIFTS

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Abstract. In this paper, we introduce a 3-Brownian shift $T_{\sigma,\theta}$ on the Hilbert space $H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}$, which is a natural extension of the classical Brownian shift $B_{\sigma,\theta}$ on $H^2(\mathbb{D}) \oplus \mathbb{C}$. This is motivated by Brownian extensions in the context of 3-isometries recently developed by A. Crăciunescu and L. Suciu. We investigate the problem of unitary equivalence for 3-Brownian shifts on invariant subspaces of the type $\mathcal{M}_0 \oplus \mathcal{M}_1$, where $\mathcal{M}_0 \subseteq H^2(\mathbb{D}^2)$ and $\mathcal{M}_1 \subseteq H^2(\mathbb{D}) \oplus \mathbb{C}$. Here, $\mathcal{M}_1$ turns out to be an invariant subspace of the respective Brownian shift $B_{\sigma,\theta}$. We also study the asymptotic behaviour of the normalized 3-Brownian shifts. This work is motivated by Richter [12] and very recently by work on Brownian shift on $H^2(\mathbb{D}) \oplus \mathbb{C}$ in [6].

1. Introduction

Let $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{T}$ denote the complex plane, the open unit disk, and the unit circle, respectively. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For $V \in \mathcal{B}(\mathcal{H})$, the subspaces $\mathcal{N}(V)$ and $\mathcal{R}(V)$ denote the kernel (null space) and the range of V, respectively. A closed subspace $M \subseteq \mathcal{H}$ is said to be *invariant* under $T \in \mathcal{B}(\mathcal{H})$ if $T(M) \subseteq M$.

In this paper, we consider the Hardy spaces of the unit disk and the unit bidisk, denoted by $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$, respectively. The Hardy space of the unit disk is defined by

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n\geq 0} a_n z^n : \sum_{n\geq 0} |a_n|^2 < \infty \right\},$$

with inner product

$$\langle f, g\rangle = \sum_{n\geq 0} a_n \overline{b_n},$$

where $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$.

The Hardy space of the unit bidisk is given by

$$H^2(\mathbb{D}^2) = \left\{ F(z_1, z_2) = \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n : \sum_{m,n\geq 0} |a_{m,n}|^2 < \infty \right\},$$

equipped with the corresponding ℓ^2 coefficient norm.

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We will also need vector-valued Hardy spaces. For a Hilbert space E, let $H^2(\mathbb{D}; E)$ denote the space of E-valued analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients.

Given the Hilbert spaces E and F, we denote by $H^\infty(\mathbb{D}; \mathcal{B}(E, F))$ the Banach space of bounded analytic $\mathcal{B}(E, F)$-valued functions on $\mathbb{D}$, where $\mathcal{B}(E, F)$ denotes the space of bounded linear operators from E to F. In the scalar case, when $E = F = \mathbb{C}$, we simply write $H^\infty(\mathbb{D})$.

For each $\varphi \in H^\infty(\mathbb{D}; \mathcal{B}(E, F))$, the multiplication operator M_φ is defined by

$$M_\varphi : H^2(\mathbb{D}; E) \to H^2(\mathbb{D}; F), \qquad (M_\varphi f)(z) = \varphi(z) f(z).$$

A function $\varphi \in H^\infty(\mathbb{D}; \mathcal{B}(E, F))$ is called *inner* if M_φ is an isometry. Equivalently, the radial boundary values satisfy that $\varphi(e^{it})$ is an isometry from E to F for almost every $t \in [0, 2\pi)$.

For such an inner function φ, the associated model space is given by

$$K_\varphi := H^2(\mathbb{D}; F) \ominus \varphi H^2(\mathbb{D}; E).$$

Classically, a Brownian shift in the Hilbert space $H^2(\mathbb{D}) \oplus \mathbb{C}$ is defined by

$$B_{\sigma, e^{i\theta}} = \begin{bmatrix} S & \sigma(1 \otimes 1) \\ 0 & e^{i\theta} \end{bmatrix},$$

where S denotes the unilateral shift on $H^2(\mathbb{D})$, $\sigma > 0$ is a covariance parameter, and $\theta \in [0, 2\pi)$ ([1, Definition 5.5]). The operator $1 \otimes 1 : \mathbb{C} \to H^2(\mathbb{D})$ is defined by assigning to each scalar $\alpha \in \mathbb{C}$ the constant function on $\mathbb{D}$ with the value α, i.e.,

$$(1 \otimes 1)(\alpha)(z) = \alpha, \quad \text{for all } z \in \mathbb{D}.$$

The Brownian shift $B_{\sigma,\theta}$ is a special case of Brownian unitaries, also referred to as B-unitary ([3, Definition 1.1]), which plays a special role when studying 2-isometries ([1, Section 5], also see [11, 14]).

Motivated by Brownian unitaries in the context of 2-isometries, Crăciunescu and Suciu recently developed the following 3-Brownian unitaries in the context of 3-isometries [5]. Given a Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2,$$

an operator $B \in \mathcal{B}(\mathcal{H})$ is called a *3-Brownian unitary* if, with respect to this decomposition, it has the block matrix representation

$$B = \begin{bmatrix} V_0 & \sigma E_0 & 0 \\ 0 & V_1 & \sigma E_1 \\ 0 & 0 & U \end{bmatrix},$$

where V_j, E_j are isometries satisfying

$$\mathcal{N}(V_j^*) = \mathcal{R}(E_j), \qquad j = 0, 1,$$

U is unitary, and $\sigma > 0$.

The Brownian shift $B_{\sigma,\theta}$ first appears as part of studying time shift operators ([1, Section 5]) and has become an important member of the larger class called B-operator which turns out to be a very useful object in understanding several properties in operator theory [4]. Motivated by this, we introduce a 3-Brownian shift on $H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}$. A 3-*Brownian shift* is a 3-Brownian unitary on $H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}$ of the following form

$$T_{\sigma,\theta} = \begin{bmatrix} M_{z_1} & \sigma J & 0 \\ 0 & S & \sigma(1 \otimes 1) \\ 0 & 0 & e^{i\theta} \end{bmatrix},$$

where M_{z_1} denotes multiplication by z_1 on $H^2(\mathbb{D}^2)$, S is the unilateral shift on $H^2(\mathbb{D})$, $J : H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is the canonical isometric embedding defined by $(Jf)(z_1, z_2) = f(z_2)$, $\sigma > 0$ and $\theta \in [0, 2\pi)$. The operator $T_{\sigma,\theta}$ can be viewed as the Brownian shift analogue in the setting of 3-isometries.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two invariant subspaces of T_{σ_1,θ_1} and T_{σ_2,θ_2}, respectively. We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *unitarily equivalent*, and write $T_{\sigma_1,\theta_1}\big|_{\mathcal{M}_1} \cong T_{\sigma_2,\theta_2}\big|_{\mathcal{M}_2}$, if there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that

$$U\, T_{\sigma_1,\theta_1}\big|_{\mathcal{M}_1} = T_{\sigma_2,\theta_2}\big|_{\mathcal{M}_2} U.$$

Motivated by the results of Richter [12] and more recently by the work of Das et al. [6], we investigate the problem of unitary equivalence of 3-Brownian shifts when restricted to certain lifted type invariant subspaces (cf. [8, 9]). In the classical case of the Hardy space of the unit disk, the famous Beurling theorem yields that all nonzero invariant subspaces of $H^2(\mathbb{D})$ are unitarily equivalent. In the polydisk case, this was investigated in [2] (also see [13]).

A fundamental structural feature of the operator $T_{\sigma,\theta}$ is that its invariant subspaces necessarily project to invariant subspaces of the lower 2×2 Brownian shift. More precisely, if

$$P : \mathcal{H} \to H^2(\mathbb{D}) \oplus \mathbb{C}, \qquad P(F, f, \alpha) = (f, \alpha),$$

denotes the canonical projection and

$$B_{\sigma,e^{i\theta}}(f, \alpha) = (zf + \sigma\alpha,\ e^{i\theta}\alpha)$$

is the lower Brownian shift, then one has the intertwining relation

$$PT_{\sigma,\theta} = B_{\sigma,e^{i\theta}} P.$$

As a consequence, for any $T_{\sigma,\theta}$-invariant subspace $\mathcal{M} \subset \mathcal{H}$, the projected subspace $P(\mathcal{M})$ is automatically invariant for $B_{\sigma,e^{i\theta}}$. This perspective justifies our focus on invariant subspaces of $T_{\sigma,\theta}$ whose lower components are prescribed to be of certain types, namely Type I or Type II (to be defined below).

This observation has an important conceptual implication. The invariant subspace structure of the classical Brownian shift is already completely understood, and its invariant subspaces fall into well-known families (Type I and Type II in the sense of Agler–Stankus [1, p. 21]). Since every invariant subspace of $T_{\sigma,\theta}$ must project onto one of these known subspaces, there is no additional freedom at the level of the lower block. In [1], Agler and Stankus present the following description of the invariant subspaces for Brownian shifts. If $\mathscr{M}$ is a nonzero closed subspace of $H^2(\mathbb{D}) \oplus \mathbb{C}$, then $\mathscr{M}$ is invariant under $B_{\sigma,e^{i\theta}}$ if and only if it has one of the following forms:

$$\mathscr{M} = \varphi H^2(\mathbb{D}) \oplus \{0\} \quad (\textit{Type I}), \tag{1.1}$$

for some inner function $\varphi \in H^\infty(\mathbb{D})$, or

$$\mathscr{M} = \mathbb{C}\begin{bmatrix} g \\ 1 \end{bmatrix} \oplus \big(\varphi H^2(\mathbb{D}) \oplus \{0\}\big) \quad (\textit{Type II}), \tag{1.2}$$

for some inner function $\varphi \in H^\infty(\mathbb{D})$ such that $\varphi(e^{i\theta})$ exists, and

$$g(z) = \sigma \cdot \frac{\overline{\varphi(e^{i\theta})}\,\varphi(z) - 1}{z - e^{i\theta}}, \qquad z \in \mathbb{D}.$$

It is also worth noting that $g \perp \varphi H^2(\mathbb{D})$ (see [1, p. 23]).

Note that in order to describe the invariant subspaces of $T_{\sigma,\theta}$ of the form

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \end{bmatrix},$$

where $\mathcal{M}_1$ is a nontrivial invariant subspace of the corresponding Brownian shift $B_{\sigma,\theta}$, it suffices to determine $\mathcal{M}_0$. In the next section, we will deduce that $\mathcal{M}_0$ has the following two choices:

(a) If $\mathcal{M}_1$ is of Type I (see (1.1)) for some inner function $\varphi \in H^\infty(\mathbb{D})$, then

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \;\oplus\; \Psi H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}})$$

for some Hilbert space $\mathcal{E}_{\mathcal{M}}$, and an operator-valued inner function $\Psi \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_\varphi)\big)$. Here $\varphi_{z_2}(z_1, z_2) := \varphi(z_2)$. We call the respective $\mathcal{M}$ as *lifted Type I invariant subspace* of $T_{\sigma,\theta}$.

(b) If $\mathcal{M}_1$ is of Type II (see (1.2)) for some inner function $\varphi \in H^\infty(\mathbb{D})$, then

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi\, H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}})$$

for some Hilbert space $\mathcal{E}_{\mathcal{M}}$, and an operator-valued inner function $\Psi \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_\varphi)\big)$ satisfying $\Psi(0)\mathcal{E}_{\mathcal{M}} \subset g^\perp \subset K_\varphi$. Here $H^2_{z_1}(\mathbb{D}) := i_{z_1}(H^2(\mathbb{D}))$ is a closed subspace of $H^2(\mathbb{D}^2)$ where $i_{z_1} : H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is an isometry defined by $(i_{z_1} f)(z_1, z_2) := f(z_1)$, $f \in H^2(\mathbb{D})$. The notation g_{z_2} is defined by $g_{z_2}(z_1, z_2) := g(z_2)$ We call the respective $\mathcal{M}$ as *lifted Type II invariant subspace* of $T_{\sigma,\theta}$. As $g \in K_\varphi$, the orthogonal complement $g^\perp$ is taken in K_φ.

Let us also consider the trivial case. Observe that if $\mathcal{M}_1 = \{0\}$, then a necessary and sufficient condition for $\mathcal{M}_0 \oplus \{0\}$ to be invariant under $T_{\sigma,\theta}$ is that $\mathcal{M}_0$ be an M_{z_1}-invariant subspace of $H^2(\mathbb{D}^2)$. Moreover, two such invariant subspaces are unitarily equivalent if and only if their first components are unitarily equivalent with respect to M_{z_1}. This situation can be completely described by the Beurling–Lax–Halmos theorem [10, Theorem 2.1, p. 239].

We now state our main result of this paper.

Theorem 1.1. *Let* $\theta_1, \theta_2 \in [0, 2\pi)$ *and* $\sigma_1, \sigma_2 > 0$. *Let*

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \end{bmatrix}, \qquad \mathcal{N} = \begin{bmatrix} \mathcal{N}_0 \\ \mathcal{N}_1 \end{bmatrix}$$

be nonzero closed invariant subspaces of lifted Type I or Type II for the 3*-Brownian shifts* T_{σ_1,θ_1} *and* T_{σ_2,θ_2}, *respectively. Then*

$$T_{\sigma_1,\theta_1}\big|_{\mathcal{M}} \cong T_{\sigma_2,\theta_2}\big|_{\mathcal{N}}$$

if and only if $\sigma_1 = \sigma_2$, $\dim \mathcal{E}_{\mathcal{M}} = \dim \mathcal{E}_{\mathcal{N}}$ *and one of the following holds:*

(i) Both $\mathcal{M}$ and $\mathcal{N}$ are of lifted Type I.

(ii) Both $\mathcal{M}$ and $\mathcal{N}$ are of lifted Type II, where

$$\mathcal{M}_0 = \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g^{(1)}_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}),$$
$$\mathcal{N}_0 = \varphi^{(2)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g^{(2)}_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}}),$$

and

$$\mathcal{M}_1 = \mathbb{C} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} \oplus \big(\varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\}\big),$$
$$\mathcal{N}_1 = \mathbb{C} \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix} \oplus \big(\varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\}\big),$$

with

$$g_j(z) = \sigma_j \cdot \frac{\overline{\varphi^{(j)}(e^{i\theta_j})}\, \varphi^{(j)}(z) - 1}{z - e^{i\theta_j}}, \qquad j = 1, 2,$$

for some inner function $\varphi^{(j)} \in H^\infty(\mathbb{D})$ and for some operator-valued inner functions

$$\Psi^{(1)} \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_{\varphi^{(1)}})\big) \text{ with } \Psi(0)\mathcal{E}_{\mathcal{M}} \subset {g^{(1)}}^{\perp} \subset K_{\varphi^{(1)}}, \tag{A}$$

$$\Psi^{(2)} \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_{\varphi^{(2)}})\big) \text{ with } \Psi(0)\mathcal{E}_{\mathcal{N}} \subset {g^{(2)}}^{\perp} \subset K_{\varphi^{(2)}}. \tag{B}$$

such that

$$\theta_1 = \theta_2, \qquad \|g^{(1)}\| = \|g^{(2)}\|,$$

Remark 1.2. The conditions (A) and (B) ensure that the decomposition of $\mathcal{M}_0$ and $\mathcal{N}_0$ in Theorem 1.1(ii) is orthogonal, which is crucial for our result. For example, if we choose $\mathcal{E}_{\mathcal{M}}$ to be $K_{\varphi^{(1)}}$ and $\psi^{(1)}(z) = I$ where I is the identity operator, then the components $g^{(1)}_{z_2} H^2_{z_1}(\mathbb{D})$ and $H^2(\mathbb{D}; K_{\varphi^{(1)}})$ are not orthogonal as $g^{(1)} \in K_{\varphi^{(1)}}$.

We now relate our analysis to the asymptotic property of Brownian shifts. Recall that if T is a contraction on a Hilbert space $\mathcal{H}$, then T is said to be *pure*, denoted $T \in C_{\cdot 0}$, if

$$\text{SOT-}\lim_{m\to\infty} T^{*m} = 0,$$

where SOT denotes the strong operator topology. Moreover, T is said to belong to the class C_{00} if both T and T^* are pure.

Given $T \in B(\mathcal{H})$ is said to be *power bounded* if the sequence $\{\|T^n\|\}_{n=0}^\infty$ of real numbers is bounded. It is known that the classical Brownian shift $B_{\sigma,\theta}$ is not power-bounded. In particular, it is not similar to a contraction. Nevertheless, after a suitable normalization, $B_{\sigma,\theta}$ belongs to the class C_{00} [6, Section 3].

This behaviour serves as a guiding principle in our setting. In fact, using the corresponding property of $B_{\sigma,\theta}$, we first show that the 3-Brownian shift $T_{\sigma,\theta}$ is also not power-bounded, and hence not similar to a contraction. We then prove that an analogous asymptotic phenomenon persists: for every $\sigma > 0$ and $\theta \in [0, 2\pi)$,

$$\frac{1}{\|T_{\sigma,e^{i\theta}}\|} T_{\sigma,e^{i\theta}} \in C_{00}.$$

Note that the operator $T_{\sigma,e^{i\theta}}$ admits a decomposition as a perturbation of an isometry. Indeed,

$$T_{\sigma,e^{i\theta}} = T_s + R,$$

where

$$T_s = \begin{bmatrix} M_{z_1} & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an isometry, and

$$R = \begin{bmatrix} 0 & \sigma J & 0 \\ 0 & 0 & \sigma(1 \otimes 1) \\ 0 & 0 & e^{i\theta} - 1 \end{bmatrix}.$$

This is where it differs from the classical Brownian shift, as in contrast to the classical case, the perturbation R is no longer of rank one ([6, P. 87]). Indeed, due to the presence of the operator J, the perturbation is of infinite rank. Thus, $T_{\sigma,e^{i\theta}}$ may be regarded as an infinite-rank perturbation of the isometry T_s.

1.1. **Plan of the paper.** In Section 2, we describe invariant subspaces of $T_{\sigma,\theta}$ of the form $\mathcal{M}_0 \oplus \mathcal{M}_1$, where $\mathcal{M}_0 \subseteq H^2(\mathbb{D}^2)$ and $\mathcal{M}_1 \subseteq H^2(\mathbb{D}) \oplus \mathbb{C}$. This description is then used to identify the unitary operators that arise in the study of unitary equivalence between such invariant subspaces. We also show that there exist invariant subspaces of $T_{\sigma,\theta}$ which are not of this form; in particular, this is demonstrated by proving that the vector $(0,0,1)^t$ is not cyclic for $T_{\sigma,\theta}$.

In Section 3, we establish several necessary conditions for two invariant subspaces to be unitarily equivalent. In Section 4, we prove our main result, Theorem 1.1, whose proof relies on Lemma 4.1, providing structural constraints on the corresponding unitary intertwiner.

Finally, in Section 5, we show that a normalized $T_{\sigma,\theta}$ belongs to the class C_{00}, in analogy with the classical Brownian shift.

2. Lifted Type Invariant subspaces of $T_{\sigma,\theta}$

In this section, we present a description of invariant subspaces of $T_{\sigma,\theta}$ which we call *lifted type invariant subspaces* arising from the invariant subspaces of the classical Brownian shift. These spaces are of the type

$$\mathcal{M} := \begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \end{bmatrix} \subseteq \begin{bmatrix} H^2(\mathbb{D}^2) \\ H^2(\mathbb{D}) \\ \mathbb{C} \end{bmatrix}, \qquad \mathcal{M}_0 \subseteq H^2(\mathbb{D}^2),\ \mathcal{M}_1 \subseteq \begin{bmatrix} H^2(\mathbb{D}) \\ \mathbb{C} \end{bmatrix}.$$

Here $\mathcal{M}_1 \subseteq H^2(\mathbb{D}) \oplus \mathbb{C}$ is an invariant subspace of $B_{\sigma,\theta}$. As noted in the introduction, $\mathcal{M}_1$ has two choices, namely Type I and Type II subspaces.

We now present a description of $\mathcal{M}$. To do this, we identify

$$H^2(\mathbb{D}^2) \cong H^2\big(\mathbb{D}; H^2(\mathbb{D})\big)$$

with respect to the variable z_1. Under this identification, M_{z_1} becomes the unilateral shift on the vector-valued Hardy space $H^2(\mathbb{D}; H^2(\mathbb{D}))$.

For an inner function $\varphi \in H^\infty(\mathbb{D})$, we have the following orthogonal decomposition

$$H^2(\mathbb{D}) = \varphi H^2(\mathbb{D}) \oplus K_\varphi, \qquad K_\varphi := H^2(\mathbb{D}) \ominus \varphi H^2(\mathbb{D}).$$

Consequently,

$$H^2(\mathbb{D}^2) = H^2\big(\mathbb{D}; \varphi H^2(\mathbb{D})\big) \oplus H^2\big(\mathbb{D}; K_\varphi\big),$$

and the first summand coincides with $\varphi_{z_2} H^2(\mathbb{D}^2)$. For any one variable function f, the notation f_{z_2} is fixed for a function in two variable defined by $f_{z_2}(z_1, z_2) := f(z_2)$.

The following theorem presents a description of invariant subspaces of $T_{\sigma,\theta}$ of the form $\begin{bmatrix}\mathcal{M}_0\\ \mathcal{M}_1\end{bmatrix}$.

Theorem 2.1. *Let $\sigma > 0$ and $\theta \in [0, 2\pi)$. Let $T_{\sigma,\theta}$ be a 3-Brownian shift and $B_{\sigma,\theta}$ be the Brownian shift. Let $\mathcal{M} = \begin{bmatrix}\mathcal{M}_0\\ \mathcal{M}_1\end{bmatrix}$ be an invariant subspace of $T_{\sigma,\theta}$, where $\mathcal{M}_1$ is a nonzero invariant subspace of $B_{\sigma,\theta}$. Then $\mathcal{M}_0$ must be one of the following forms:*

(i) If $\mathcal{M}_1 = \begin{bmatrix}\varphi H^2\\ 0\end{bmatrix}$, where $\varphi \in H^\infty(\mathbb{D})$, is an inner function, then there exists a Hilbert space $\mathcal{E}_\mathcal{M}$ and an operator-valued inner function $\Psi \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_\mathcal{M}, K_\varphi)\big)$ such that

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \oplus \Psi H^2(\mathbb{D}; \mathcal{E}_\mathcal{M}).$$

(ii) If $\mathcal{M}_1 = \mathbb{C}\begin{bmatrix}g\\ 1\end{bmatrix} \oplus \big(\varphi H^2 \oplus \{0\}\big)$, where $\varphi \in H^\infty(\mathbb{D})$ is an inner function such that $\varphi(e^{i\theta})$ exists, and

$$g(z) = \sigma\left(\frac{\overline{\varphi(e^{i\theta})}\,\varphi(z) - 1}{z - e^{i\theta}}\right), \qquad z \in \mathbb{D},$$

then there exist a Hilbert space $\mathcal{E}_\mathcal{M}$ and an operator-valued inner function $\Psi \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_\mathcal{M}, K_\varphi)\big)$ such that

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi\, H^2(\mathbb{D}; \mathcal{E}_\mathcal{M}),$$

with $\Psi(0)\mathcal{E}_\mathcal{M} \subset g^\perp \subset K_\varphi$.

Proof. Since $T_{\sigma,\theta}(\mathcal{M}) \subseteq \mathcal{M}$, it follows that $\mathcal{M}_0$ must be invariant under multiplication by z_1. Note that we have only two nontrivial choices for $\mathcal{M}_1$, which we divide into the following cases.

(i) In this case, we take $\mathcal{M}_1$ to be

$$\begin{bmatrix}\varphi H^2(\mathbb{D})\\ 0\end{bmatrix},$$

where $\varphi \in H^\infty(\mathbb{D})$ is an inner function. From

$$T_{\sigma,\theta}\begin{bmatrix}0\\ \varphi f\\ 0\end{bmatrix} \in \mathcal{M} = \begin{bmatrix}\mathcal{M}_0\\ \mathcal{M}_1\end{bmatrix},$$

we have $\varphi_{z_2} f_{z_2} \in \mathcal{M}_0$ for every $f \in H^2(\mathbb{D})$. Since $\mathcal{M}_0$ is M_{z_1}-invariant, we obtain $\varphi_{z_2} H^2(\mathbb{D}^2) \subseteq \mathcal{M}_0$.

To get a nice description of $\mathcal{M}_0$, we will use the Beurling–Lax–Halmos theorem [10, Theorem 2.1, p. 239]. Since $\varphi_{z_2} H^2(\mathbb{D}^2) \subset \mathcal{M}_0$, it follows that $\mathcal{M}_0$ decomposes orthogonally as

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \oplus \mathcal{N}_0, \qquad \mathcal{N}_0 := \mathcal{M}_0 \cap H^2\big(\mathbb{D}; K_\varphi\big).$$

The subspace $\mathcal{N}_0$ is closed and invariant under M_{z_1}, now viewed as the shift on $H^2(\mathbb{D}; K_\varphi)$. By the Beurling–Lax–Halmos theorem, there exists a Hilbert space $\mathcal{E}_\mathcal{M}$ and an operator-valued inner function

$$\Psi \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_\mathcal{M}, K_\varphi)\big)$$

such that

$$\mathcal{N}_0 = \Psi H^2(\mathbb{D}; \mathcal{E}_\mathcal{M}).$$

Therefore,

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \oplus \Psi H^2(\mathbb{D}; \mathcal{E}_\mathcal{M}).$$

(ii) In this case, we take the remaining choice for $\mathcal{M}_1$, i.e,

$$\mathcal{M}_1 = \mathbb{C}\begin{pmatrix} g \\ 1 \end{pmatrix} \oplus (\varphi H^2(\mathbb{D}) \oplus \{0\}),$$

where $\varphi \in H^2(\mathbb{D})$ is inner with $\varphi(e^{i\theta})$ defined and

$$g(z) = \sigma \cdot \frac{\varphi(e^{i\theta})\varphi(z) - 1}{z - e^{i\theta}}, \quad z \in \mathbb{D}.$$

In addition to $\varphi_{z_2} H^2(\mathbb{D}^2) \subseteq \mathcal{M}_0$, we also have $g_{z_2} \in \mathcal{M}_0$ in this case. Since $\mathcal{M}_0$ is z_1-invariant, we obtain $z_1^i g_{z_2} \in \mathcal{M}_0$, $i \geq 0$. Thus $g_{z_2} H^2_{z_1}(\mathbb{D}) \subseteq \mathcal{M}_0$. Note that we have the orthogonal decomposition

$$H^2(\mathbb{D}^2) = \varphi_{z_2} H^2(\mathbb{D}^2) \ \oplus \ H^2(\mathbb{D}; K_\varphi).$$

Since the first summand above already lies in $\mathcal{M}_0$, it follows that

$$\mathcal{M}_0 = \varphi_{z_2} H^2(\mathbb{D}^2) \ \oplus \ \mathcal{N}_0, \qquad \mathcal{N}_0 := \mathcal{M}_0 \cap H^2(\mathbb{D}; K_\varphi),$$

where $\mathcal{N}_0$ is a closed M_{z_1}-invariant subspace of $H^2(\mathbb{D}; K_\varphi)$. We also know that $g \perp \varphi H^2(\mathbb{D})$, which further implies that $g_{z_2} \perp \varphi_{z_2} H^2(\mathbb{D}^2)$, and thus $g_{z_2} \in \mathcal{N}_0$.

Consider the wandering subspace of $\mathcal{N}_0$ with respect to M_{z_1},

$$W := \mathcal{N}_0 \ominus z_1 \mathcal{N}_0.$$

By the Wold decomposition for the unilateral shift,

$$\mathcal{N}_0 = \bigoplus_{n \geq 0} z_1^n W. \tag{2.1}$$

Since g_{z_2} is independent of z_1, we necessarily have $g_{z_2} \perp z_1 \mathcal{N}_0$, hence $g_{z_2} \in W$. Consequently,

$$W = \operatorname{span}\{g_{z_2}\} \oplus W', \qquad W' := W \ominus \operatorname{span}\{g_{z_2}\}.$$

From (2.1), we have

$$\mathcal{N}_0 = g_{z_2} H^2_{z_1}(\mathbb{D}) \ \oplus \ \Big(\bigoplus_{n \geq 0} z_1^n W'\Big).$$

Here $g_{z_2} H^2_{z_1}(\mathbb{D}) := \overline{\operatorname{span}}\{z_1^i g_{z_2};\, i \geq 0\} \subseteq H^2(\mathbb{D}^2)$.

The remaining summand

$$\mathcal{N}_0' := \bigoplus_{n\geq 0} z_1^n W' \subset H^2(\mathbb{D}; K_\varphi)$$

is again a closed M_{z_1}-invariant subspace. Hence, again by the Beurling–Lax–Halmos theorem, there exist a Hilbert space $\mathcal{E}_{\mathcal{M}}$ and an operator-valued inner function

$$\Psi \in H^\infty(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_\varphi))$$

such that

$$\mathcal{N}_0' = \Psi\, H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}).$$

Since $W' \perp g$, the wandering space satisfies $\Psi(0)\mathcal{E}_{\mathcal{M}} \subset g^\perp \subset K_\varphi$.

Combining the above decompositions, we conclude that every closed M_{z_1}-invariant subspace N that satisfies the stated assumptions admits the representation

$$N = \varphi_{z_2}\, H^2(\mathbb{D}^2) \;\oplus\; g_{z_2}\, H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi\, H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}),$$

where Ψ is an inner multiplier taking values in $K_\varphi \ominus \operatorname{span}\{g\}$. □

Remark 2.2. It follows that the minimal lifted Type I invariant subspace of $T_{\sigma,\theta}$ for which $\varphi H^2(\mathbb{D}) \oplus \{0\}$, is invariant under $B_{\sigma,\theta}$, is given by

$$\mathcal{M}_\varphi^{\mathrm{I}} = \varphi_{z_2}\, H^2(\mathbb{D}^2) \oplus \big(\varphi H^2(\mathbb{D}) \oplus \{0\}\big).$$

Similarly, the minimal lifted Type II invariant subspace for which

$$\mathbb{C}\begin{bmatrix} g \\ 1 \end{bmatrix} \oplus \big(\varphi H^2(\mathbb{D}) \oplus \{0\}\big)$$

is invariant under $B_{\sigma,\theta}$, is

$$\mathcal{M}_\varphi^{\mathrm{II}} := N_{\min} \oplus \left(\mathbb{C}\begin{bmatrix} g \\ 1 \end{bmatrix} \oplus \big(\varphi H^2(\mathbb{D}) \oplus \{0\}\big)\right),$$

where

$$N_{\min} = \varphi_{z_2}\, H^2(\mathbb{D}^2) \;\oplus\; g_{z_2}\, H^2_{z_1}(\mathbb{D}).$$

We note that characterizing invariant subspaces for $T_{\sigma,\theta}$ appears to be a subtle problem. Indeed, there are invariant subspaces as cyclic subspaces of the form

$$\mathscr{C}_x := \overline{\operatorname{span}}\{T^n_{\sigma,\theta} x : n \geq 0\}, \qquad x \in H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}.$$

It can be seen from the different choices of x that the invariant subspace $\mathscr{C}_x$ does not agree with any of the defined types. For example, consider the vector $x = (0,0,1)^t$. For the Brownian shift $B_{\sigma,\theta}$ the vector $(0,1)^t$ is cyclic, that is,

$$\overline{\operatorname{span}}\{B^n_{\sigma,\theta}(0,1)^t : n \geq 0\} = H^2(\mathbb{D}) \oplus \mathbb{C}.$$

However, this is not true for $T_{\sigma,\theta}$, if we take the orbit of the vector $(0,0,1)^t$ under $T_{\sigma,\theta}$. The following proposition illustrates that the structure of invariant subspaces generated by single vectors can be quite complicated. For this reason, we restrict our attention to tractable classes of invariant subspaces of $T_{\sigma,\theta}$.

Proposition 2.3 (Non-cyclicity of the vector $(0,0,1)^t$)**.** *Let $T_{\sigma,\theta}$ be a 3-Brownian shift on $\mathcal{H} = H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}$, where $\sigma > 0, \theta \in [0, 2\pi)$. Let $e_3 = (0,0,1)^t$. Then e_3 is not cyclic for $T_{\sigma,\theta}$, that is,*

$$\mathcal{M} := \overline{\operatorname{span}}\{T_{\sigma,\theta}^n e_3 : \ n \geq 0\} \neq \mathcal{H}.$$

Proof. We first show that for every $n \geq 0$,

$$T_{\sigma,\theta}^n e_3 = \Big(\sigma^2 \sum_{k=0}^{n-2} e^{ik\theta}\, h_{n-2-k}(z_1, z_2),\ \sigma \sum_{k=0}^{n-1} e^{ik\theta} z^{\,n-1-k},\ e^{in\theta}\Big)^t, \tag{2.2}$$

where

$$h_m(z_1, z_2) = \sum_{\substack{a,b \geq 0 \\ a+b=m}} z_1^a z_2^b, \qquad m \in \mathbb{Z},$$

with the convention that $h_m \equiv 0$ for $m < 0$. We argue by induction on n.

For $n = 0$ is trivial and for $n = 1$, we have

$$T_{\sigma,\theta} e_3 = (0, \sigma, e^{i\theta})^t,$$

which agrees with (2.2).

Assume that the formula holds for some $n \geq 1$, i.e.

$$T_{\sigma,\theta}^n e_3 = (F_n, f_n, e^{in\theta})^t,$$

with

$$F_n = \sigma^2 \sum_{k=0}^{n-2} e^{ik\theta}\, h_{n-2-k}, \qquad f_n = \sigma \sum_{k=0}^{n-1} e^{ik\theta} z^{n-1-k}.$$

Applying $T_{\sigma,\theta}$ yields

$$T_{\sigma,\theta}^{n+1} e_3 = \big(z_1 F_n + \sigma f_n(z_2),\ z f_n + \sigma e^{in\theta},\ e^{i(n+1)\theta}\big)^t.$$

Note that the third coordinate matches with the induction step. For the middle coordinate,

$$z f_n + \sigma e^{in\theta} = \sigma \sum_{k=0}^{n-1} e^{ik\theta} z^{n-k} + \sigma e^{in\theta} = \sigma \sum_{k=0}^{n} e^{ik\theta} z^{n-k}.$$

For the first coordinate, we compute

$$z_1 F_n + \sigma f_n(z_2) = \sigma^2 \sum_{k=0}^{n-2} e^{ik\theta} z_1 h_{n-2-k} + \sigma^2 \sum_{k=0}^{n-1} e^{ik\theta} z_2^{n-1-k}.$$

Using the identity $h_{m+1} = z_1 h_m + z_2^{m+1}$ for $m \geq 0$, we obtain

$$z_1 F_n + \sigma f_n(z_2) = \sigma^2 \sum_{k=0}^{n-1} e^{ik\theta} h_{n-1-k}.$$

This establishes the claimed formula for $n+1$ and completes the induction.

Now since a finite linear combination of symmetric polynomials is symmetric, also the pointwise limit of symmetric polynomials is symmetric, it follows that the first coordinate of vectors in $\mathcal{M}$ is always symmetric in z_1, z_2. Hence, $\mathcal{M} \neq \mathcal{H}$. $\square$

3. Auxillary Results

To investigate the unitary equivalence problem for 3-Brownian shifts $T_{\sigma,\theta}$, when restricted to lifted Type I and lifted Type II invariant subspaces, we first compute the operator norm of $T_{\sigma,\theta}$ restricted to these invariant subspaces. Since the operator norm is preserved under unitary equivalence, it provides a useful criterion for distinguishing non-unitarily equivalence.

Lemma 3.1. *Let $T_{\sigma,\theta}$ be the 3–Brownian shift acting on $\mathcal{H} = H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}) \oplus \mathbb{C}$ by*

$$T_{\sigma,\theta}(F, f, \alpha)^t = \big(z_1 F + \sigma f_{z_2},\ zf + \sigma\alpha,\ e^{i\theta}\alpha\big)^t, \qquad \sigma > 0.$$

Let $\mathcal{M}$ be one of lifted Type I or lifted Type II invariant subspace of $T_{\sigma,\theta}$. Then $\|T_{\sigma,\theta}\big|_{\mathcal{M}}\| = \sqrt{1+\sigma^2}$.

Proof. We first consider the action of $T_{\sigma,\theta}$ on the full space $\mathcal{H}$. In $H^2(\mathbb{D}^2)$, the subspaces $z_1 H^2(\mathbb{D}^2)$ and $\{g(z_2) : g \in H^2(\mathbb{D})\}$ are orthogonal, and multiplication by z_1 and by z act isometrically on $H^2(\mathbb{D}^2)$ and $H^2(\mathbb{D})$, respectively. Consequently, for $(F, f, \alpha)^t \in \mathcal{H}$,

$$\|T_{\sigma,\theta}(F, f, \alpha)^t\|^2 = \|F\|^2 + (\sigma^2+1)\|f\|^2 + (\sigma^2+1)|\alpha|^2 \le (1+\sigma^2)\|(F, f, \alpha)\|^2.$$

Thus $\|T_{\sigma,\theta}\| \le \sqrt{1+\sigma^2}$. Since the equality is attained if we take vectors supported in the second or third coordinates, we obtain $\|T_{\sigma,\theta}\big|_{\mathcal{M}}\| = \sqrt{1+\sigma^2}$. This completes the proof. □

The following lemma provides an immediate necessary condition for the unitary equivalence of invariant subspaces of 3-Brownian shifts.

Lemma 3.2. *Let $\mathcal{M}_1$ and M_2 be lifted Type I or lifted Type II invariant subspaces of T_{σ_1,θ_1} and T_{σ_2,θ_2}, respectively. Assume that $T_{\sigma_1,\theta_1}\big|_{M_1} \cong T_{\sigma_2,\theta_2}\big|_{\mathcal{M}_2}$. Then $\sigma_1 = \sigma_2$.*

Proof. The conclusion follows directly from Lemma 3.1. □

We now record the following elementary fact for a unitary equivalent shift-invariant subspaces of a vector-valued Hardy space (see the proof of [7, Theorem 3.1(i)]). We add the proof for the sake of completeness.

Proposition 3.3. *Let $\psi_i \in H^\infty(\mathbb{D}; B(\mathcal{E}_i, E))$, $i = 1, 2$, be operator–valued inner functions. Then M_z-invariant subspaces $\psi_1 H^2(\mathbb{D}; \mathcal{E}_1)$ and $\psi_2 H^2(\mathbb{D}; \mathcal{E}_2)$ are unitarily equivalent if and only if $\dim \mathcal{E}_1 = \dim \mathcal{E}_2$.*

Proof. Let $N_i := \psi_i H^2(\mathbb{D}; \mathcal{E}_i) \subset H^2(\mathbb{D}; E)$ for $i = 1, 2$. Since ψ_i is inner, the multiplication operator $M_{\psi_i} : H^2(\mathbb{D}; \mathcal{E}_i) \to H^2(\mathbb{D}; E)$ is an isometry, and N_i is invariant under M_z.

First, note that $N_i \ominus zN_i = \psi_i\big(H^2(\mathbb{D}; \mathcal{E}_i) \ominus zH^2(\mathbb{D}; \mathcal{E}_i)\big)$. Since $H^2(\mathbb{D}; \mathcal{E}_i) \ominus zH^2(\mathbb{D}; \mathcal{E}_i)$ consists of constant functions, we obtain $N_i \ominus zN_i = \psi_i \mathcal{E}_i$, and therefore

$$\dim(N_i \ominus zN_i) = \dim \mathcal{E}_i.$$

($\Rightarrow$) Suppose that N_1 and N_2 are unitarily equivalent with respect to M_z. Then there exists a unitary operator $U : \mathcal{N}_1 \to N_2$ such that $UM_z|_{N_1} =$

$M_z|_{N_2}U$. Hence $U(zN_1) = zN_2$, and consequently $U(N_1 \ominus zN_1) = N_2 \ominus zN_2$. Thus

$$\dim \mathcal{E}_1 = \dim(N_1 \ominus zN_1) = \dim(N_2 \ominus zN_2) = \dim \mathcal{E}_2.$$

($\Leftarrow$) Conversely, assume $\dim \mathcal{E}_1 = \dim \mathcal{E}_2$. Then there exists a unitary $V : \mathcal{E}_1 \to \mathcal{E}_2$. Using the orthogonal Wold decompositions

$$N_i = \bigoplus_{n \geq 0} z^n (N_i \ominus zN_i) = \bigoplus_{n \geq 0} z^n \psi_i \mathcal{E}_i,$$

define $U : N_1 \to N_2$ on the dense algebraic sum by

$$U\left(\sum_{n \geq 0} z^n \psi_1 e_n\right) = \sum_{n \geq 0} z^n \psi_2 (V e_n), \qquad e_n \in \mathcal{E}_1.$$

Since the summands $z^n \psi_i \mathcal{E}_i$ are mutually orthogonal and ψ_i acts isometrically on $\mathcal{E}_i$, the map U is a well-defined unitary operator. By construction, $UM_z = M_z U$, so N_1 and N_2 are unitarily equivalent with respect to M_z. □

Unitary equivalence and compressions. In the next section, we repeatedly pass from an operator to its compression to a closed subspace. Since much of our analysis relies on unitary equivalence, it is useful to record the following straightforward but important permanence property: unitary equivalence is preserved under compression, provided the unitary operator maps the relevant subspaces onto one another.

Proposition 3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$, and let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator such that*

$$UT_1 = T_2 U.$$

Let $\mathcal{K}_1 \subset \mathcal{H}_1$ and $\mathcal{K}_2 \subset \mathcal{H}_2$ be closed subspaces with $U(\mathcal{K}_1) = \mathcal{K}_2$. Define the compressions

$$B_1 := P_{\mathcal{K}_1} T_1\big|_{\mathcal{K}_1} \in \mathcal{B}(\mathcal{K}_1), \qquad B_2 := P_{\mathcal{K}_2} T_2\big|_{\mathcal{K}_2} \in \mathcal{B}(\mathcal{K}_2).$$

Then the restriction $U|_{\mathcal{K}_1} : \mathcal{K}_1 \to \mathcal{K}_2$ is unitary and intertwines the compressions, namely

$$\left(U|_{\mathcal{K}_1}\right) B_1 = B_2 \left(U|_{\mathcal{K}_1}\right).$$

Proof. Since $U(\mathcal{K}_1) = \mathcal{K}_2$, one has

$$UP_{\mathcal{K}_1} = P_{\mathcal{K}_2} U. \tag{3.1}$$

For $x \in \mathcal{K}_1$, using (3.1) and $UT_1 = T_2 U$, we compute

$$UB_1 x = UP_{\mathcal{K}_1} T_1 x = P_{\mathcal{K}_2} UT_1 x = P_{\mathcal{K}_2} T_2 U x = B_2 U x,$$

where the last equality uses $Ux \in \mathcal{K}_2$. This proves the desired intertwining relation on $\mathcal{K}_1$. □

4. Unitary Equivalence

In this section, we prove our main result, Theorem 1.1.

For convenience, we fix the following notation for this section. Let $\mathcal{M} = \begin{bmatrix}\mathcal{M}_0\\ \mathcal{M}_1\end{bmatrix}$, $\mathcal{N} = \begin{bmatrix}\mathcal{N}_0\\ \mathcal{N}_1\end{bmatrix}$, be invariant subspace of 3-Brownian shit of lifted type.

(i) If $\mathcal{M}, \mathcal{N}$, are of lifted Type I, then

$$\mathcal{M}_0 = \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; \Psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}),$$
$$\mathcal{N}_0 = \varphi^{(2)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; \Psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}}),$$

and

$$\mathcal{M}_1 = \varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\},$$
$$\mathcal{N}_1 = \varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\},$$

for some inner function $\varphi^{(j)} \in H^\infty(\mathbb{D})$ and for some operator-valued inner functions $\Psi^{(j)} \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_{\varphi^{(j)}})\big)$.

(ii) If $\mathcal{M}, \mathcal{N}$, are of lifted Type II, then

$$\mathcal{M}_0 = \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g^{(1)}_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}),$$
$$\mathcal{N}_0 = \varphi^{(2)}_{z_2} H^2(\mathbb{D}^2) \;\oplus\; g^{(2)}_{z_2} H^2_{z_1}(\mathbb{D}) \;\oplus\; \Psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}}),$$

and

$$\mathcal{M}_1 = \mathbb{C} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} \oplus \big(\varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\}\big),$$
$$\mathcal{N}_1 = \mathbb{C} \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix} \oplus \big(\varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\}\big),$$

with

$$g^{(j)} = \sigma_j \cdot \frac{\overline{\varphi^{(j)}(e^{i\theta_j})}\,\varphi^{(j)} - 1}{z - e^{i\theta_j}}, \qquad j = 1, 2,$$

for some inner function $\varphi^{(j)} \in H^\infty(\mathbb{D})$ and for some operator-valued inner functions

$$\Psi^{(1)} \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_{\varphi^{(1)}})\big), \text{ with } \Psi(0)\mathcal{E}_{\mathcal{M}} \subset {g^{(1)}}^{\perp} \subset K_{\varphi^{(1)}},$$

$$\Psi^{(2)} \in H^\infty\big(\mathbb{D}; \mathcal{B}(\mathcal{E}_{\mathcal{M}}, K_{\varphi^{(2)}})\big), \text{ with } \Psi(0)\mathcal{E}_{\mathcal{N}} \subset {g^{(2)}}^{\perp} \subset K_{\varphi^{(2)}}.$$

In the following lemma, we provide necessary conditions for the existence of a unitary intertwiner between lifted-type invariant subspaces. In particular, such an operator must preserve the coordinate decomposition and act diagonally, inducing a shift-commuting unitary on the Hardy component.

Lemma 4.1. *Let* $\mathcal{M} = \begin{bmatrix}\mathcal{M}_0\\ \mathcal{M}_1\end{bmatrix}$, $\mathcal{N} = \begin{bmatrix}\mathcal{N}_0\\ \mathcal{N}_1\end{bmatrix}$ *be lifted type invariant subspaces of* T_{σ,θ_1} *and* T_{σ,θ_2}*, respectively. Suppose that there exists a unitary operator* $U : \mathcal{M} \to \mathcal{N}$ *such that*

$$UT_{\sigma,\theta_1}|_{\mathcal{M}} = T_{\sigma,\theta_2}|_{\mathcal{N}} U.$$

Then the following assertions hold.

(i) *The unitary operator U preserves the coordinate decomposition and hence admits the block representation*

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix},$$

where $U_1 : \mathcal{M}_0 \to \mathcal{N}_0$ and $U_2 : \mathcal{M}_1 \to \mathcal{N}_1$ are unitary operators.

(ii) *The operator U_1 commutes with M_{z_1}, i.e. , $U_1(z_1 F) = z_1 U_1(F)$, for all $F \in \mathcal{M}_0$.*

(iii) *$\mathcal{M}$, and $\mathcal{N}$ must be of the lifted type. The unitary operator U_2 maps $\varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\}$ onto $\varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\}$ via*

$$U_2 \begin{bmatrix} \varphi^{(1)} f \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda\, \varphi^{(2)} f \\ 0 \end{bmatrix}, \qquad f \in H^2(\mathbb{D}),$$

for some $\lambda \in \mathbb{T}$. Accordingly, we shall identify U_2 with a unitary map from $\varphi^{(1)} H^2(\mathbb{D})$ onto $\varphi^{(2)} H^2(\mathbb{D})$. Moreover, in the Type II case, U_2 maps span $\begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix}$ *onto* span $\begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix}$, *i.e.,*

$$U_2 \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} = \beta \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix}$$

for some $\beta \in \mathbb{C} \setminus \{0\}$.

(iv) *Considering the isometric embedding $J : H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$, defined by $J(h)(z_1, z_2) = h(z_2)$, we have*

$$U_1\big(J(G)\big) = J\big(U_2 G\big), \qquad G \in \varphi^{(1)} H^2(\mathbb{D}),$$

and $U_1(Jg^{(1)}) = \beta J g^{(2)}$ for some $\beta \in \mathbb{C} \setminus \{0\}$.

(v) $\dim \mathcal{E}_{\mathcal{M}}$ *and* $\dim \mathcal{E}_{\mathcal{N}}$ *are equal.*

Proof. (*i*) Let $F \in \mathcal{M}_0$ and

$$U \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} G \\ g \\ c \end{bmatrix} \in \begin{bmatrix} \mathcal{N}_0 \\ \mathcal{N}_1 \end{bmatrix}, \tag{4.1}$$

where $G \in \mathcal{N}_0$ and $\begin{bmatrix} g \\ c \end{bmatrix} \in \mathcal{N}_1$. Since $UT_{\sigma,\theta_1}|_{\mathcal{M}} = T_{\sigma,\theta_2}|_{\mathcal{N}} U$, we have

$$U \begin{bmatrix} z_1 F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 G + \sigma g_{z_2} \\ zg + c\sigma \\ ce^{i\theta_2} \end{bmatrix}.$$

Computing the norm of both sides, we get

$$\|F\|^2 = \|G\|^2 + (\sigma^2 + 1)(\|g\|^2 + |c|^2). \tag{4.2}$$

By (4.1), we have $\|F\|^2 = \|G\|^2 + \|g\|^2 + |c|^2$. This together with (4.2) yields that $g = 0$ and $c = 0$. We obtain $U(F, 0, 0)^t = (U_1 F, 0, 0)^t$ for some isometry $U_1 : \mathcal{M}_0 \to \mathcal{N}_0$. Applying the same argument to U^{-1} shows that U_1 is onto. Hence, U_1 is unitary. A similar argument works in the case $\begin{bmatrix} 0 \\ \mathcal{M}_1 \end{bmatrix}$. Thus we have the required unitary operators $U_1 : \mathcal{M}_0 \to \mathcal{N}_0$ and $U_2 : \mathcal{M}_1 \to \mathcal{N}_1$.

(*ii*) The intertwining relation $T_{\sigma,\theta_1}|_{\mathcal{M}} U = UT_{\sigma,\theta_2}|_{\mathcal{N}}$ applied to $(F, 0, 0)^t$ gives

$$U_1(z_1 F) = z_1 U_1(F), \qquad F \in \mathcal{M}_0.$$

(iii) Note that if a vector

$$\begin{bmatrix} 0 \\ G \\ 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \end{bmatrix},$$

then G must be in $\varphi^{(1)} H^2(\mathbb{D})$. Let

$$U \begin{bmatrix} 0 \\ \varphi^{(1)} f \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ g \\ c \end{bmatrix}, \quad f \in H^2(\mathbb{D}), \begin{bmatrix} g \\ c \end{bmatrix} \in \mathcal{M}_1. \tag{4.3}$$

Then

$$T_{\sigma,\theta_2} U \begin{bmatrix} 0 \\ \varphi^{(1)} f \\ 0 \end{bmatrix} = T_{\sigma,\theta_2} \begin{bmatrix} 0 \\ g \\ c \end{bmatrix}.$$

By the intertwining relation $UT_{\sigma,\theta_1}|_{\mathcal{M}} = T_{\sigma,\theta_2}|_{\mathcal{N}} U$, we have

$$UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ \varphi^{(1)} f \\ 0 \end{bmatrix} = U \begin{bmatrix} \sigma \varphi_{1_{z_2}} f_{z_2} \\ z\varphi_1 f \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma g_{z_2} \\ zg + \sigma c \\ ce^{i\theta_2} \end{bmatrix} = T_{\sigma,\theta_2} \begin{bmatrix} 0 \\ g \\ c \end{bmatrix}.$$

But this yields that $U_1(\sigma \varphi_{1_{z_2}} f_{z_2}) = \sigma g_{z_2}$. Thus $\|f\| = \|g\|$. From (4.3), we obtain $c = 0$ and U_2 maps $\varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\}$ onto $\varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\}$. Furthermore, since U_2 is unitary, U_2 must map the orthogonal complement of $\varphi^{(1)} H^2(\mathbb{D}) \oplus \{0\}$ in $\mathcal{M}_1$ to the orthogonal complement of $\varphi^{(2)} H^2(\mathbb{D}) \oplus \{0\}$ in $\mathcal{N}_1$. Thus, we can assume that in the Type II case, $U_2 \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} = \beta \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix}$ for some $\beta \in \mathbb{C}$. This shows that $\mathcal{M}$, $\mathcal{N}$ must be of the same type.

By comparing the second coordinate in

$$UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ \varphi^{(1)} \\ 0 \end{bmatrix} = T_{\sigma,\theta_2}\big|_{\mathcal{N}} U \begin{bmatrix} 0 \\ \varphi^{(1)} \\ 0 \end{bmatrix},$$

we obtain

$$U_2(z\varphi^{(1)}) = z\, U_2(\varphi^{(1)}).$$

It follows that

$$U_2(\varphi^{(1)} H^2(\mathbb{D})) = U_2(\varphi^{(1)}) H^2(\mathbb{D}).$$

Since $U_2(\varphi^{(1)} H^2(\mathbb{D})) = \varphi^{(2)} H^2(\mathbb{D})$, we obtain $U_2(\varphi^{(1)}) = \lambda \varphi^{(2)}$ for some $|\lambda| = 1$.

(iv) This is straightforward from the relation $UT_{\sigma,\theta_1}|_{\mathcal{M}} = T_{\sigma,\theta_2}|_{\mathcal{N}} U$ applied on the vector $(0, G, 0)^t$, $G \in \varphi^{(1)} H^2(\mathbb{D})$.

(v) Assume that $\mathcal{M}$ and $\mathcal{N}$ are of Type I. By part (i), the unitary U_1 maps $\mathcal{M}_0$ onto $\mathcal{N}_0$. Moreover, by parts (iii) and (iv), U_1 maps $\varphi^{(1)}_{z_2} H^2(\mathbb{D}^2)$ onto $\varphi^{(2)}_{z_2} H^2(\mathbb{D}^2)$. Consequently, U_1 must map

$$\Psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}) \quad \text{onto} \quad \Psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}}).$$

By part (ii), we note that $U_1(z_1 F) = z_1 U_1(F)$ for $F \in \mathcal{M}_0$. Since U_1 intertwines M_{z_1}, the subspaces

$$\Psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}) \quad \text{and} \quad \Psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}})$$

are unitarily equivalent M_{z_1}-invariant subspaces. Hence, by the Proposition 3.3 theorem,

$$\dim \mathcal{E}_{\mathcal{M}} = \dim \mathcal{E}_{\mathcal{N}}.$$

An analogous argument yields the same conclusion in the Type II case. □

Remark 4.2. From part (iv) of Lemma 4.1, we observe that certain information carried by U_2 is reflected in U_1. This motivated a natural strategy to decompose the first coordinate into several components, according to the structural relationship between U_1 and U_2.

Let us now proceed to prove our main theorem, which we restate here for convenience.

Theorem 4.3. *Let $\theta_1, \theta_2 \in [0, 2\pi)$ and $\sigma_1, \sigma_2 > 0$. Let*

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \end{bmatrix}, \qquad \mathcal{N} = \begin{bmatrix} \mathcal{N}_0 \\ \mathcal{N}_1 \end{bmatrix}$$

be nonzero closed invariant subspaces of lifted Type I or Type II for the 3-*Brownian shifts T_{σ_1,θ_1} and T_{σ_2,θ_2}, respectively. Then*

$$T_{\sigma_1,\theta_1}\big|_{\mathcal{M}} \cong T_{\sigma_2,\theta_2}\big|_{\mathcal{N}}$$

if and only if $\sigma_1 = \sigma_2$, $\dim \mathcal{E}_{\mathcal{M}} = \dim \mathcal{E}_{\mathcal{N}}$ and one of the following holds:

(i) Both $\mathcal{M}$ and $\mathcal{N}$ are of lifted Type I.

(ii) Both $\mathcal{M}$ and $\mathcal{N}$ are of lifted Type II, $\theta_1 = \theta_2$, and $\|g^{(1)}\| = \|g^{(2)}\|$.

Proof. To show the necessity part, assume that

$$T_{\sigma_1,\theta_1}\big|_{\mathcal{M}} \cong T_{\sigma_2,\theta_2}\big|_{\mathcal{N}},$$

and let $U : \mathcal{M} \to \mathcal{N}$ be a unitary operator such that $UT_{\sigma_1,\theta_1}\big|_{\mathcal{M}} = T_{\sigma_2,\theta_2}\big|_{\mathcal{N}}U$. From Lemma 3.2, $\sigma_1 = \sigma_2$. By Lemma 4.1(iii), the invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ are of the same type and by Lemma 4.1(v), $\dim \mathcal{E}_{\mathcal{M}} = \dim \mathcal{E}_{\mathcal{N}}$.

Furthermore, by Lemma 4.1(i), the intertwining unitary U decomposes as

$$U = U_1 \oplus U_2 \quad \text{from } \mathcal{M}_0 \oplus \mathcal{M}_1 \text{ onto } \mathcal{N}_0 \oplus \mathcal{N}_1.$$

In addition, Proposition 3.4 yields

$$U_2\, B_{\sigma,\theta_1}\big|_{\mathcal{M}_1} = B_{\sigma,\theta_2}\big|_{\mathcal{N}_1} U_2.$$

It now follows from the unitary equivalence theory for Brownian shifts [6, Theorem 2.1] that, if $\mathcal{M}_1$ and $\mathcal{N}_1$ are of Type II, then

$$\theta_1 = \theta_2 \quad \text{and} \quad \|g^{(1)}\| = \|g^{(2)}\|.$$

Now to see the sufficiency part, assume that $\sigma := \sigma_1 = \sigma_2$. We divide the proof into the following two cases according to the type of $\mathcal{M}, \mathcal{N}$:
(i) $\mathcal{M}, \mathcal{N}$ are of lifted Type I. Since $\dim \mathcal{E}_{\mathcal{M}} = \dim \mathcal{E}_{\mathcal{N}}$, take a unitary $V : \mathcal{E}_{\mathcal{M}} \to \mathcal{E}_{\mathcal{N}}$ and define a unitary operator $V_1 : \psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}) \to \psi^{(2)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{N}})$ by

$$V_1 \left(\sum_{i=0}^{\infty} z^i \psi^{(1)} v_i \right) = \sum_{i=0}^{\infty} z^i \psi^{(2)} V v_i, \quad v_i \in \mathcal{E}_{\mathcal{M}}.$$

Clearly V_1 commutes with M_z. Now define a natural choice of unitary operators

$U_1 : \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2) \to \varphi^{(2)}_{z_2} H^2(\mathbb{D}),\ \ U_2 : \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2) \oplus \{0\} \to \varphi^{(2)}_{z_2} H^2(\mathbb{D}^2) \oplus \{0\}$

by

$$U_1(\varphi^{(1)}_{z_2} G) = \varphi^{(2)}_{z_2} G, \qquad U_2 \begin{bmatrix} \varphi^{(1)} f \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi^{(2)} f \\ 0 \end{bmatrix}, \quad G \in H^2(\mathbb{D}^2),\ f \in H^2(\mathbb{D}).$$

In this case, the unitary $U : \mathcal{M} \to \mathcal{N}$, is defined by

$$U \begin{bmatrix} F_1 + F_2 \\ f \\ 0 \end{bmatrix} = \begin{bmatrix} U_1 F_1 + V F_2 \\ U_2 \begin{bmatrix} f \\ 0 \end{bmatrix} \end{bmatrix},$$

where $F_1 \in \varphi^{(1)}_{z_2} H^2(\mathbb{D}^2)$, $F_2 \in \psi^{(1)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}})$, $f \in \varphi^{(1)} H^2(\mathbb{D}^2)$. Now it is easy to verify that $UT_{\sigma,\theta_1}|_{\mathcal{M}} = T_{\sigma,\theta_2}|_{\mathcal{N}} U$. Indeed, it suffices to verify the intertwining on vectors of the form

$$\begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix},\ F \in M_0, \ \text{ and } \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix},\ f \in \varphi^{(1)} H^2(\mathbb{D}).$$

It is easy to verify the intertwining relation for the first vector using Lemma 4.1(ii). To verify this for the second vector, write $f = \varphi^{(1)} h$, $h \in H^2(\mathbb{D})$. Then

$$\begin{aligned} UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix} &= U \begin{bmatrix} \sigma J f \\ S f \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma U_1 J f \\ U_2 \begin{bmatrix} Sf \\ 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \sigma U_1 (\varphi^{(1)}_{z_2} h) \\ U_2 B_{\sigma,\theta_1} \begin{bmatrix} f \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sigma \varphi^{(2)}_{z_2} h \\ B_{\sigma,\theta_2} U_2 \begin{bmatrix} f \\ 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \sigma J \varphi^{(2)} h \\ B_{\sigma,\theta_2} U_2 \begin{bmatrix} f \\ 0 \end{bmatrix} \end{bmatrix} = T_{\sigma,\theta_2} U \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}. \end{aligned}$$

(ii) $\mathcal{M}, \mathcal{N}$ are of lifted Type II. The components

$$\varphi^{(i)}_{z_2} H^2(\mathbb{D}^2), \Psi^{(i)} H^2(\mathbb{D}; \mathcal{E}_{\mathcal{M}}) \quad \text{and} \quad \varphi^{(i)} H^2(\mathbb{D})$$

are handled exactly as in part (i). Thus, it remains only to define unitary maps on the additional summands.

Define a linear map

$$\widetilde{U} : \operatorname{span} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} \longrightarrow \operatorname{span} \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix}$$

by

$$\widetilde{U} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} = \begin{bmatrix} g^{(2)} \\ 1 \end{bmatrix}.$$

Since $\|g^{(1)}\| = \|g^{(2)}\|$, this map is unitary.

Next, define

$$\widetilde{V} : g^{(1)}_{z_2} H^2_{z_1}(\mathbb{D}) \longrightarrow g^{(2)}_{z_2} H^2_{z_1}(\mathbb{D})$$

by

$$\widetilde{V}(z_1^k g_{z_2}^{(1)}) = z_1^k g_{z_2}^{(2)}, \qquad k \geq 0.$$

Now extend it by linearity and continuity to all of $g_{z_2}^{(1)} H^2_{z_1}(\mathbb{D})$. Then $\widetilde{V}$ is unitary and commutes with M_{z_1}.

Combining these maps with the unitary operators constructed in part (i), we obtain a unitary $U : \mathcal{M} \to \mathcal{N}$ such that

$$UT_{\sigma,\theta_1}\big|_{\mathcal{M}} = T_{\sigma,\theta_2}\big|_{\mathcal{N}} U.$$

To verify this it suffices to check the intertwining relation on the vector

$$\begin{bmatrix} g^{(1)} f \\ 0 \\ 0 \end{bmatrix}, \quad f \in H^2_{z_1}(\mathbb{D}) \text{ and } \quad \begin{bmatrix} 0 \\ g_{z_2}^{(1)} \\ 1 \end{bmatrix}.$$

On the first vector, it is easy to verify, so we verify on the second vector. By definition,

$$T_{\sigma,\theta_1} \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma J g^{(1)} \\ S g^{(1)} + \sigma \\ e^{i\theta_1} \end{bmatrix}.$$

Applying U, we obtain

$$UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma \widetilde{V}(J g^{(1)}) \\ \widetilde{U} \begin{bmatrix} S g^{(1)} + \sigma \\ e^{i\theta_1} \end{bmatrix} \end{bmatrix}.$$

Now, by the definition of $\widetilde{V}$,

$$\widetilde{V}(J g^{(1)}) = J g^{(2)},$$

and since $\widetilde{U}$ intertwines the restricted Brownian shifts on $\mathcal{M}_1$ and $\mathcal{N}_1$, i.e,

$$\widetilde{U}\, B_{\sigma,\theta_1}\big|_{\mathcal{M}_1} = B_{\sigma,\theta_2}\big|_{\mathcal{N}_1} \widetilde{U},$$

we obtain

$$\widetilde{U} \begin{bmatrix} S g^{(1)} + \sigma \\ e^{i\theta_1} \end{bmatrix} = \widetilde{U}\, B_{\sigma,\theta_1} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} = B_{\sigma,\theta_2}\, \widetilde{U} \begin{bmatrix} g^{(1)} \\ 1 \end{bmatrix} = \begin{bmatrix} S g^{(2)} + \sigma \\ e^{i\theta_2} \end{bmatrix}.$$

Therefore,

$$UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma J g^{(2)} \\ S g^{(2)} + \sigma \\ e^{i\theta_2} \end{bmatrix} = T_{\sigma,\theta_2} \begin{bmatrix} 0 \\ g^{(2)} \\ 1 \end{bmatrix} = T_{\sigma,\theta_2} U \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix}.$$

Hence,

$$UT_{\sigma,\theta_1} \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix} = T_{\sigma,\theta_2} U \begin{bmatrix} 0 \\ g^{(1)} \\ 1 \end{bmatrix}.$$

This completes the proof in the lifted Type II case. □

5. $\frac{1}{\|T_{\sigma,\theta}\|}T_{\sigma,\theta}$ BELONGS TO THE CLASS C_{00}

In this section, we show that a 3-Brownian shift $T_{\sigma,\theta}$ does not belong to C_{00} by showing that it is not power bounded. Then we prove that the normalized 3-Brownian shift $\frac{1}{\sqrt{1+\sigma^2}}T_{\sigma,\theta}$ belongs to the class C_{00}. The results and techniques are analogous to the results obtained for Brownian shifts in [6, Section 3].

Proposition 5.1. *For $\sigma > 0, \theta \in [0, 2\pi)$, the operator $T_{\sigma,\theta}$ is not power bounded.*

Proof. This follows from

$$\|T^n_{\sigma,\theta}\|^2 \geq \|T^n_{\sigma,\theta}(0,0,1)\|^2 \geq \|B^n_{\sigma,\theta}(0,1)\|^2 \geq 1+n\sigma^2, \quad n\geq 0. \qquad \square$$

The following is the main result of this section.

Theorem 5.2. *For $\sigma > 0, \theta \in [0, 2\pi)$, let $\widetilde{T}_{\sigma,\theta} := \frac{1}{\sqrt{1+\sigma^2}}T_{\sigma,\theta}$. Then*

$$\widetilde{T}_{\sigma,\theta} \in C_{00}.$$

Proof. Let $\widetilde{T} \equiv \widetilde{T}_{\sigma,\theta}$. We will show that $\widetilde{T}^n \to 0$ and $\widetilde{T}^{*n} \to 0$ in the strong operator topology. To show $\widetilde{T}^n \to 0$ in SOT, take $u = (F, f, \alpha)^t \in \mathcal{H}$. We have

$$\|T^n_{\sigma,\theta}u\| \leq \|T^n_{\sigma,\theta}(F,0,0)^t\| + \|T^n_{\sigma,\theta}(0,f,0)^t\| + \|T^n_{\sigma,\theta}(0,0,\alpha)^t\|, \quad n\in\mathbb{Z}_+.$$

The first term equals $\|F\|$, since M_{z_1} is an isometry. A direct computation as above shows for $n \in \mathbb{Z}_+$,

$$\|T^n_{\sigma,\theta}(0,f,0)^t\|^2 \leq (1+\sigma^2 n)\|f\|^2, \qquad \|T^n_{\sigma,\theta}(0,0,\alpha)^t\|^2 \leq C_\sigma(1+n^2)|\alpha|^2,$$

for some constant $C_\sigma > 0$. Consequently,

$$\|\widetilde{T}^n u\| \leq \frac{\|F\| + \sqrt{1+\sigma^2 n}\,\|f\| + \sqrt{C_\sigma(1+n^2)}\,|\alpha|}{(1+\sigma^2)^{n/2}} \longrightarrow 0 \quad (n\to\infty),$$

since the exponential decay dominates the polynomial growth. Hence, $\widetilde{T}^n \to 0$ in SOT.

We now prove that $\widetilde{T}^{*n} \to 0$ in SOT. Since $\|\widetilde{T}\| \leq 1$, it suffices to verify the convergence on a dense subset of $\mathcal{H} = H^2(\mathbb{D}^2)\oplus H^2(\mathbb{D})\oplus\mathbb{C}$. Recall that

$$T_{\sigma,\theta} = \begin{bmatrix} M_{z_1} & \sigma J & 0 \\ 0 & S & \sigma(1\otimes 1) \\ 0 & 0 & e^{i\theta}\end{bmatrix}, \qquad \widetilde{T}_{\sigma,\theta} = \frac{1}{\sqrt{1+\sigma^2}}T_{\sigma,\theta},$$

where $J : H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is given by $(Jf)(z_1, z_2) = f(z_2)$. The adjoint of T has the block form

$$T^* = \begin{bmatrix} M^*_{z_1} & 0 & 0 \\ \sigma J^* & S^* & 0 \\ 0 & \sigma(1\otimes 1)^* & e^{-i\theta}\end{bmatrix}.$$

Note that

$$J^*(z_1^m z_2^k) = \begin{cases} z^k, & m = 0, \\ 0, & m\geq 1, \end{cases} \qquad (1\otimes 1)^*(f) = \langle f, 1\rangle,$$

and that $M_{z_1}^*$ and S^* are the backward shifts on $H^2(\mathbb{D}^2)$ and $H^2(\mathbb{D})$, respectively.

Since finite linear combinations of monomials are dense, it suffices to check strong convergence on the following orthonormal basis vectors:

$$(z_1^m z_2^k, 0, 0)^t, \qquad (0, z^k, 0)^t, \qquad (0, 0, 1)^t.$$

Case 1. $x = (0, 0, 1)^t$. Then $T^{*n}x = (0, 0, e^{-in\theta})^t$, and hence

$$\|\widetilde{T}^{*n}x\| = (1+\sigma^2)^{-n/2} \longrightarrow 0 \quad (n \to \infty).$$

Case 2. $x = (0, z^k, 0)^t$, $k \geq 0$. If $k \geq 1$, then

$$T^*(0, z^k, 0)^t = (0, z^{k-1}, 0)^t,$$

while

$$T^*(0, 1, 0)^t = (0, 0, \sigma)^t.$$

Consequently, after at most $k+1$ iterations, the vector $(0, z^k, 0)$ enters the scalar component $\mathbb{C}$, and thereafter remains bounded in norm by $\max\{1, \sigma\}$. Therefore, for all n,

$$\|T^{*n}(0, z^k, 0)^t\| \leq C_k$$

for some constant C_k independent of n. It follows that

$$\|\widetilde{T}^{*n}(0, z^k, 0)^t\| \leq C_k(1+\sigma^2)^{-n/2} \longrightarrow 0 \quad (n \to \infty).$$

Case 3. $x = (z_1^m z_2^k, 0, 0)^t$, $m, k \geq 0$. If $m \geq 1$, then

$$T^*(z_1^m z_2^k, 0, 0)^t = (z_1^{m-1} z_2^k, 0, 0)^t,$$

and hence

$$T^{*m}(z_1^m z_2^k, 0, 0)^t = (z_2^k, 0, 0)^t.$$

Applying T^* once more yields

$$T^*(z_2^k, 0, 0)^t = (0, \sigma z^k, 0)^t.$$

As in Case 2, after finitely many additional iterations the vector enters the scalar component and remains uniformly bounded. Thus there exists a constant $C_{m,k} > 0$ such that

$$\|T^{*n}(z_1^m z_2^k, 0, 0)^t\| \leq C_{m,k} \quad \text{for all } n \geq 1.$$

Consequently,

$$\|\widetilde{T}^{*n}(z_1^m z_2^k, 0, 0)^t\| \leq C_{m,k}(1+\sigma^2)^{-n/2} \longrightarrow 0 \quad (n \to \infty).$$

Combining the above cases, we conclude that $\widetilde{T}^{*n}x \to 0$ for all basis vectors x. Since $\sup_n \|\widetilde{T}^{*n}\| \leq 1$ and these vectors span a dense subspace of $\mathcal{H}$, it follows that $\widetilde{T}^{*n} \xrightarrow{\text{SOT}} 0$. This completes the proof. □

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