Moment theory approach to Gaussian Quadratures with prescribed nodes

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(joint work with Aljaž Zalar)

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Atomic measures

For $x \in \mathbb{R}$, δ_x stands for the Dirac measure supported on x.
Finitely atomic positive measure on $\mathbb{R}$:

$$\mu = \sum_{j=1}^{\ell} \rho_j \delta_{x_j}, \quad \rho_j > 0, x_j \in \mathbb{R}.$$

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The **evaluation at ∞** is the linear functional, defined by

$$\mathbf{ev}_\infty : \mathbb{R}[\mathrm{x}]_{\leq D} \to \mathbb{R}, \quad \sum_{k=0}^{D} f_k \mathrm{x}^k \mapsto f_D.$$

Allowing $\mathbf{ev}_\infty$ to be a part of a finitely atomic positive measure μ, we call μ a **generalized measure**.

Gaussian quadrature

Definition : Suppose D is a positive integer and μ is a measure on $\mathbb{R}$ whose moments up to degree D exist and are finite. A **quadrature rule** of degree D for μ is a finite set $N \subset \mathbb{R}$ together with $w : N \to \mathbb{R}_{>0}$ with

$$\int f \, d\mu = \sum_{x \in N} w(x) \, \mathrm{ev}_x(f) \quad \text{for all } \; f \in \mathbb{R}[t]_{\leq D}.$$

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A Quadrature rule for μ which contains the smallest number of atoms is called Gaussian quadrature rule (GQR).

Truncated moment sequence

A real sequence

$$\gamma \equiv \gamma^{(D)} = (\gamma_0, \gamma_1, \dots, \gamma_D) \in \mathbb{R}^{D+1}$$

of degree D, $D \in \mathbb{N} \cup \{0\}$, is said to be $\mathbb{R}-$ **truncated moment sequence** if there exists a positive Borel measure μ on $\mathbb{R}$, such that

$$\gamma_i = \int_{\mathbb{R}} x^i d\mu, \quad 0 \leq i \leq D.$$

If such a measure exists, we say that γ has a $\mathbb{R}$**–representing measure ($\mathbb{R}$–rm)**.

Moment matrix

Let $D = 2d$, $d \in \mathbb{N}$. For $\ell \in \mathbb{N}$, $\ell \leq d$ the Hankel matrix of size $(\ell+1) \times (\ell+1)$ corresponding to the sequence γ is defined by

$$M_\ell := (\gamma_{i+j-2})_{i,j=1}^{\ell+1} = \begin{array}{c c} & \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^\ell \end{array} \\ \begin{array}{c} \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^\ell \end{array} & \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_\ell \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{\ell+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2\ell-1} \\ \gamma_\ell & \gamma_{\ell+1} & \cdots & \gamma_{2\ell-1} & \gamma_{2\ell} \end{bmatrix} \end{array}$$

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and is called the ℓ**–th truncated moment matrix** of γ.For $i, j \in \mathbb{N} \cup \{0\}$, $i \leq d$, $j \leq d$, let

$$\mathbf{v}_i^{(j)} := (\gamma_{i+r-1})_{1 \leq r \leq j+1} \in \mathbb{R}^{j+1}.$$

Using this notation, we have that

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$$p(X) = a_0 \mathit{1} + \sum_{i=1}^{\ell} a_i X^i,$$

where $\mathit{1}$ and X^i represent the columns of M_ℓ indexed by these monomials. Then $p(X)$ is a vector from the linear span of the columns of M_ℓ.

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The **rank** of γ, denoted by $\operatorname{rank}\gamma$, is defined by

$$\operatorname{rank}\gamma = \begin{cases} d+1, & \text{if } M_d \text{ is nonsingular,} \\ \min\left\{ i \colon \mathbf{v}_i^{(d)} \in \operatorname{span}\{\mathbf{v}_0^{(d)}, \ldots, \mathbf{v}_{i-1}^{(d)}\} \right\}, & \text{if } M_d \text{ is singular.} \end{cases}$$

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1. M_{r-1} is positive definite.
2. If $r < d+1$, denoting

$$(\varphi_0, \ldots, \varphi_{r-1}) := M_{r-1}^{-1}\mathbf{v}_{\mathbf{r}}^{(\mathbf{r-1})},$$

the equality

$$\gamma_j = \varphi_0\gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1}$$

holds for $j = r, \ldots, 2d$.

If γ is singular and prg, we call

$$p_{(\gamma)}(\mathrm{x}) := \mathrm{x}^r - \sum_{i=0}^{r-1} \varphi_i \mathrm{x}^i \in \mathbb{R}[\mathrm{x}],$$

with φ_i as in (2), the **generating polynomial** of γ.

Localizing moment matrices

Let $\gamma = \{\gamma_i\}_{i=0}^{D}$. The **Riesz functional** $L : \mathbb{R}[x]_{\leq D} \to \mathbb{R}$ of γ is defined by

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For $f \in \mathbb{R}[x]_{\leq D}$, an f**−localizing moment sequence** $\mathcal{H}_f$ **of** γ is

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$\mathcal{H}_f$: moment matrix of the sequence $f \cdot \gamma$ and is called a f–localizing moment matrix.

We write $\mathcal{H}_f(\ell)$ for the ℓ-th truncated moment matrix of $f \cdot \gamma$, i.e.,

$$\mathcal{H}_f(\ell) := \left(\gamma^{(f)}_{i+j-2}\right)_{i,j=1}^{\ell+1} = \begin{array}{c c} & \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^\ell \end{array} \\ \begin{array}{c} \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^\ell \end{array} & \left[\begin{array}{ccccc} \gamma^{(f)}_0 & \gamma^{(f)}_1 & \gamma^{(f)}_2 & \cdots & \gamma^{(f)}_\ell \\ \gamma^{(f)}_1 & \gamma^{(f)}_2 & \iddots & \iddots & \gamma^{(f)}_{\ell+1} \\ \gamma^{(f)}_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma^{(f)}_{2\ell-1} \\ \gamma^{(f)}_\ell & \gamma^{(f)}_{\ell+1} & \cdots & \gamma^{(f)}_{2\ell-1} & \gamma^{(f)}_{2\ell} \end{array}\right] \end{array}$$

Elementary symmetric polynomials :

Let $d_1 \in \mathbb{N}$ and

$$e_i(\mathtt{x}_1, \ldots, \mathtt{x}_{d_1}) := \sum_{1 \le j_1 < j_2 < \ldots < j_i \le d_1} \mathtt{x}_{j_1} \mathtt{x}_{j_1} \cdots \mathtt{x}_{j_i}$$

stand for the *i*-th elementary symmetric polynomial in variables $\mathtt{x}_1, \ldots, \mathtt{x}_{d_1}$. Given distinct real numbers $x_1, \ldots, x_{d_1}$, let e_i stand for $e_i(x_1, \ldots, x_{d_1})$. In particular,

$$e_0 = 1, \quad e_1 = \sum_{i=1}^{d_1} x_i, \quad e_2 = \sum_{1 \le i < j \le d_1} x_i x_j, \quad \ldots, \quad e_{d_1} = x_1 x_2 \cdots x_{d_1}. \qquad (0.0)$$

Theorem (Curto, Fialkow,1991)

Let $d \in \mathbb{N}$ and $\gamma = (\gamma_0, \ldots, \gamma_{2d}) \in \mathbb{R}^{2d+1}$ with $\gamma_0 > 0$. The following statements are equivalent :

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3. γ *is positively recursively generated.*
4. M_d *is positive semidefinite and* $\operatorname{rank} M_d = \operatorname{rank}\gamma$.
5. *One of the following statements holds :*
   1. M_d *is positive definite.*

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1. *There exists a $\mathbb{R}$–representing measure for γ.*
2. *There exists a $(\operatorname{rank}\gamma)$–atomic $\mathbb{R}$–representing measure for γ.*
3. *γ is positively recursively generated.*
4. *M_d is positive semidefinite and $\operatorname{rank} M_d = \operatorname{rank}\gamma$.*
5. *One of the following statements holds :*
    1. *M_d is positive definite.*
    2. *M_d is positive semidefinite and $\operatorname{rank} M_d = \operatorname{rank} M_{d-1}$.*

Moreover, if a $\mathbb{R}$–representing measure for γ exists, then :

(i) If $r \leq d$, then the $\mathbb{R}$–representing measure is unique and of the form $\mu = \sum_{i=1}^{r} \rho_i \delta_{x_i}$, where $x_1, \ldots, x_r$ are the roots of $p_{(\gamma)}$ and $(\rho_i)_{i=0}^{r} = V_{(x_1,\ldots,x_m)}^{-1} \mathbf{v}_0^{(r-1)}$.

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2. If $r = d + 1$, then there are infinitely many $\mathbb{R}$–representing measures for γ. All $(d+1)$–atomic ones are obtained by choosing $\gamma_{2k+1} \in \mathbb{R}$ arbitrarily, defining $\gamma_{2k+2} := (\mathbf{v}_{d+1}^{(d)})^T (M_d)^{-1} \mathbf{v}_{d+1}^{(d)}$, and using (i) for $\widetilde{\gamma} := (\gamma_0, \ldots, \gamma_{2d+1}, \gamma_{2d+2}) \in \mathbb{R}^{2k+3}$.

Problem [G. Blekherman, M. Kummer, C. Riener, M. Schweighofer, C. Vinzant]

Given $d_1, d_2 \geq 1$, a degree $D := d_1 + 2d_2 - 1$ sequence

$$\gamma^{(D)} = (\gamma_0, \gamma_1, \ldots, \gamma_D) \in \mathbb{R}^{D+1}$$

such that the moment matrix $M_{\lfloor \frac{D}{2} \rfloor}$ is positive definite and real numbers $x_1, \ldots x_{d_1}$, when does a (g)GQR with $d_1 + d_2$ nodes, containing $x_1, \ldots, x_{d_1}$, exist ?

Main Theorem

Theorem : Let $d_1, d_2 \in \mathbb{N}$ and $x_1, \ldots, x_{d_1} \in \mathbb{R}$ be distinct real numbers. Let $D := d_1 + 2d_2 - 1$. Assume that $\gamma \equiv \gamma^{(D)} = (\gamma_0, \ldots, \gamma_D) \in \mathbb{R}^{D+1}$ is a sequence such that

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$$f(\mathtt{x}) := \prod_{i=1}^{d_1} (\mathtt{x} - x_i) = \sum_{i=0}^{d_1} (-1)^i e_i \mathtt{x}^{d_1 - i}.$$

The following statements are equivalent :

1. There exists a $(d_1 + d_2)$–atomic $\mathbb{R}$–representing measure for γ with d_1 atoms equal to $x_1, \dots, x_{d_1}$.

2. The following conditions hold :

(i) The localizing matrix $\mathcal{H}_f(d_2 - 1)$ is invertible.

(ii) Denote

$$\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{d_2-1}\end{pmatrix}^T = \big(\mathcal{H}_f(d_2-1)\big)^{-1}\Big(\sum_{i=0}^{d_1}(-1)^i e_i \mathbf{v}_{d_1+d_2-i}^{(d_2-1)}\Big),$$

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let

$$h(\mathtt{x}) := f(\mathtt{x})\underbrace{\left(\mathtt{x}^{d_2} - \sum_{i=0}^{d_2-1}\lambda_i \mathtt{x}^i\right)}_{g(\mathtt{x})} = \mathtt{x}^{d_1+d_2} - \sum_{i=0}^{d_1+d_2-1}\varphi_i \mathtt{x}^i,$$

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and let

$$\widetilde{\gamma} := \{\widetilde{\gamma}_u\}_{u=0}^{D+d_1-1}$$

be the extension of γ, defined by $\widetilde{\gamma}_u = \gamma_u$ for $0 \le u \le D$ and

$$\widetilde{\gamma}_{D+\ell} = \sum_{i=1}^{d_1+d_2}\varphi_{d_1+d_2-i}\widetilde{\gamma}_{D+\ell-i} \quad \text{for} \quad \ell = 1,\ldots,d_1-1.$$

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The moment matrix $M_{d_1+d_2-1}$ of $\widetilde{\gamma}$ is positive definite.
Moreover, if the equivalent statements 1,2 hold, then the other atoms in the measure are the zeros of the polynomial $g(\mathrm{x})$.

$1 \Rightarrow 2$

We will see the proof of the theorem in the special case when $D = 5, d_1 = 2, d_2 = 2$. The moment sequence is $m = \{m_i\}_{i=0}^5$ and the corresponding moment matrix is

$$M_3 = \begin{pmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{pmatrix} > \mathbf{0}.$$

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Let $\mu = \sum_{i=1}^{4} \rho_i \delta_{x_i}$ be a representing measure of m with $x_1 = x, x_2 = y$.

$$M_5 = \begin{array}{c|cccccc|} & 1 & T & T^2 & T^3 & T^4 & T^5 \\ & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & m_0 & m_1 & m_2 & m_3 & m_4 & m_5 \\ T & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ T^2 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 \\ T^3 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 \\ T^4 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 \\ T^5 & m_5 & m_6 & m_7 & m_8 & m_9 & m_{10} \end{array}$$

The generating polynomial for M_5 is $p(t) = (t-x)(t-y)(t-x_3)(t-x_4)$.

$$\begin{aligned} p(t) &= (t^2 - (x+y)t + xy)(t^2 - (\lambda_1 t + \lambda_0)) \\ &= t^4 - (x+y)t^3 + xyt^2 - \sum_{i=0}^{1} \lambda_i (t^{2+i} - (x+y)t^{1+i} + xyt^i). \end{aligned}$$

By [CF91],

$$p(T) = 0,\ Tp(T) = 0.$$

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This yields that

- $v_4 - (x + y)v_3 + xyv_2 = \sum_{i=0}^{1} \lambda_i (v_{2+i} - (x + y)v_{1+i} + xyv_i)$,
- $v_5 - (x + y)v_4 + xyv_3 = \sum_{i=0}^{1} \lambda_i (v_{3+i} - (x + y)v_{2+i} + xyv_{i+1})$.

The generating polynomial for M_5 is $p(t) = (t-x)(t-y)(t-x_3)(t-x_4)$.

$$\begin{aligned} p(t) &= (t^2 - (x+y)t + xy)(t^2 - (\lambda_1 t + \lambda_0)) \\ &= t^4 - (x+y)t^3 + xyt^2 - \sum_{i=0}^{1} \lambda_i (t^{2+i} - (x+y)t^{1+i} + xyt^i). \end{aligned}$$

By [CF91],

$$p(T) = 0, \ Tp(T) = 0.$$

This yields that

- $v_4 - (x+y)v_3 + xyv_2 = \sum_{i=0}^{1} \lambda_i (v_{2+i} - (x+y)v_{1+i} + xyv_i)$,
- $v_5 - (x+y)v_4 + xyv_3 = \sum_{i=0}^{1} \lambda_i (v_{3+i} - (x+y)v_{2+i} + xyv_{i+1})$.

Note that

(a) $$\begin{pmatrix} v_3 - (x+y)v_2 + xy\, v_1 & v_4 - (x+y)v_3 + xy\, v_2 \end{pmatrix} = \begin{pmatrix} v_2 - (x+y)v_1 + xy\, v_0 & v_3 - (x+y)v_2 + xy\, v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

(b) $$\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix} = \begin{pmatrix} v_3 - (x+y)v_2 + xy\, v_1 & v_4 - (x+y)v_3 + xy\, v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

(b) $$\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix} = \begin{pmatrix} v_3 - (x+y)v_2 + xy\, v_1 & v_4 - (x+y)v_3 + xy\, v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix} = \begin{pmatrix} v_2 - (x+y)v_1 + xy\, v_0 & v_3 - (x+y)v_2 + xy\, v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

(b) $\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix} =$

$$\begin{pmatrix} v_3 - (x+y)v_2 + xy\, v_1 & v_4 - (x+y)v_3 + xy\, v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix} =$$
$$\begin{pmatrix} v_2 - (x+y)v_1 + xy\, v_0 & v_3 - (x+y)v_2 + xy\, v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Consider

$$\begin{pmatrix} v_4 - (x+y)v_3 + xy\, v_2 & v_5 - (x+y)v_4 + xy\, v_3 \end{pmatrix}$$

(b) $$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Consider

$$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix}$$
$$-\ (x+y) \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix}$$

(b) $$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Consider

$$\begin{aligned} &\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} \\ &\quad - (x+y) \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \\ &\qquad + xy \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \end{aligned}$$

(b) $$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Consider

$$\begin{aligned} &\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} \\ &\quad - (x+y) \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \\ &\quad + xy \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \\ &\quad = \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \end{aligned}$$

(b) $$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Using (a) we get

$$\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} = \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Consider

$$\begin{aligned} &\begin{pmatrix} v_4-(x+y)v_3+xy\,v_2 & v_5-(x+y)v_4+xy\,v_3 \end{pmatrix} \\ &\quad - (x+y) \begin{pmatrix} v_3-(x+y)v_2+xy\,v_1 & v_4-(x+y)v_3+xy\,v_2 \end{pmatrix} \\ &\quad + xy \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \\ &= \begin{pmatrix} v_2-(x+y)v_1+xy\,v_0 & v_3-(x+y)v_2+xy\,v_1 \end{pmatrix} \\ &\qquad \left[\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}^2 - (x+y)\begin{pmatrix} 0 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix} + xy\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]. \end{aligned}$$

Key step :

Let $A_1 = \begin{pmatrix} -x_1 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & 0 & -x_1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -x_2 & 1 & 0 \\ 0 & -x_2 & 1 \end{pmatrix}$.

Then

$$\begin{aligned} A_2 A_1 M_3 A_1^t A_2^t = & \left(v_4 - (x+y)v_3 + xy\, v_2 \quad v_5 - (x+y)v_4 + xy\, v_3\right) \\ & -(x+y)\left(v_3 - (x+y)v_2 + xy\, v_1 \quad v_4 - (x+y)v_3 + xy\, v_2\right) \\ & +xy\left(v_2 - (x+y)v_1 + xy\, v_0 \quad v_3 - (x+y)v_2 + xy\, v_1\right). \end{aligned}$$

$2 \Rightarrow 1$

We have

$$\tilde{\gamma} : \gamma_0, \gamma_1, \cdots, \gamma_{d_1+2d_2-1}, \gamma_{d_1+2d_2}, \cdots, \gamma_{2d_1+2d_2-2}.$$

and

$$M_{d_1+d_2-1} > 0.$$

Define $\gamma_{2d_1+2d_2-1}, \gamma_{2d_1+2d_2}$ recursively from $h(x)$.

$$M_{d_1+d_2} = \left(\begin{array}{cccc|c} & & & & \gamma_{d_1+d_2} \\ & & & & \gamma_{d_1+d_2+1} \\ & & & & \vdots \\ & & M_{d_1+d_2-1} & & \gamma_{d_1+2d_2-1} \\ & & & & \gamma_{d_1+2d_2} \\ & & & & \vdots \\ & & & & \gamma_{2d_1+2d_2+1} \\ \hline \gamma_{d_1+d_2} & \gamma_{d_1+d_2+1} & \cdots & \gamma_{2d_1+2d_2+1} & \gamma_{2d_1+2d_2+2} \end{array}\right)$$

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Thank You For Your Attention !