

The truncated univariate rational moment problem

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$$q(x) := \prod_{j=1}^p (x - \lambda_j)^{2k_j} \cdot \prod_{j=1}^r (x^2 + \eta_j)^{\ell_j},$$

where $\lambda_1, \dots, \lambda_p$ distinct real numbers η_1, \dots, η_r distinct positive real numbers.

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Take $2k := \sum_{j=1}^p 2k_j + \sum_{j=1}^r 2\ell_j$ and

$$\mathcal{R}^{(2k)} = \left\{ \frac{f}{q} : f \in \mathbb{R}[x]_{\leq 2k} \right\}.$$

Rational K -truncated moment problem

The **rational K -truncated moment problem (K -RTMP)** asks to characterize the existence of a positive Borel measure μ , supported on K , such that a linear functional $\mathcal{L} : \mathcal{R}^{(2k)} \rightarrow \mathbb{R}$ has an integral representation

$$\mathcal{L}(R) = \int_K R(x)\mu(x) \quad \forall R \in \mathcal{R}^{(2k)}.$$

If μ is such a measure, then we call it a **K -representing measure (K -rm)** for \mathcal{L} . The points λ_j (resp. $\pm i\sqrt{\eta_j}$) are called **real poles** (resp. **complex poles**).

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We denote by $\mathcal{M}_{\mathcal{L},K}^{(\text{fa})}$, $\mathcal{M}_{L,K}^{(\text{fa})}$ and $\mathcal{M}_{L,K,\Lambda}^{(\text{fa})}$ the subsets of $\mathcal{M}_{\mathcal{L},K}$, $\mathcal{M}_{L,K}$ and $\mathcal{M}_{L,K,\Lambda}$, respectively, containing all finitely atomic measures.

Example

Let $K = [0, 1]$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\mathcal{R}^{(4)} = \left\{ \frac{f}{x^2(x-1)^2} : f \in \mathbb{R}[x]_{\leq 4} \right\}$ and $\mathcal{L} : \mathcal{R}^{(4)} \rightarrow \mathbb{R}$ the linear functional defined by

$$\begin{aligned} \mathcal{L}(1) = \gamma_0^{(0)} &:= \frac{1}{48}, & \mathcal{L}\left(\frac{1}{x}\right) = \gamma_1^{(1)} &:= \frac{1}{24}, & \mathcal{L}\left(\frac{1}{x^2}\right) = \gamma_2^{(1)} &= \frac{5}{12}, \\ \mathcal{L}\left(\frac{1}{x-1}\right) = \gamma_1^{(2)} &:= -\frac{1}{24}, & \mathcal{L}\left(\frac{1}{(x-1)^2}\right) = \gamma_2^{(2)} &:= \frac{5}{12}. \end{aligned}$$

The corresponding functional $L : \mathbb{R}[x]_{\leq 4} \rightarrow \mathbb{R}$ is defined by

$$L(1) = 1, \quad L(x) = \frac{1}{2}, \quad L(x^2) = \frac{5}{12}, \quad L(x^3) = \frac{3}{8}, \quad L(x^4) = \frac{17}{48}.$$

The localizing Hankel matrices determining whether $L|_{T_{S[0,1]}^{(4)}} \geq 0$ holds are

$$\mathcal{H}_{1,\gamma} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{5}{12} \\ \frac{1}{2} & \frac{5}{12} & \frac{3}{8} \\ \frac{5}{12} & \frac{3}{8} & \frac{17}{48} \end{pmatrix}, \quad \mathcal{H}_{x,\gamma} = \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{5}{12} & \frac{3}{8} \end{pmatrix}, \quad \mathcal{H}_{1-x,\gamma} = \begin{pmatrix} \frac{1}{2} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} \end{pmatrix},$$

$$\mathcal{H}_{x(1-x),\gamma} = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{48} \end{pmatrix}.$$

They are all psd with the eigenvalues $\approx 1.54, 0.22, 0.007$ for $\mathcal{H}_{1,\gamma}$, $\approx 0.86, 0.016$ for $\mathcal{H}_{x,\gamma}$, $\approx 0.51, 0.027$ for $\mathcal{H}_{1-x,\gamma}$, and $\frac{5}{48}, 0$ for $\mathcal{H}_{x(1-x),\gamma}$. Note that $p(x) = x - \frac{1}{2}$ is a column relation of $\mathcal{H}_{x(1-x),\gamma}$ and thus the unique measure for L consists of the atoms $0, 1, \frac{1}{2}$ all with densities $\frac{1}{3}$. Hence, \mathcal{L} does not have a K -rm, even though L is K -positive.

Preordering

Given a finite set $S := \{g_1, g_2, \dots, g_n\}$ or a countable set $S := \{g_i\}_{i=1}^\infty$ in $\mathbb{R}[x]$ and $e := (e_1, \dots, e_m) \in \{0, 1\}^m$, let \underline{g}^e stand for $g_1^{e_1} g_2^{e_2} \dots g_m^{e_m}$. Let

$$E := \begin{cases} \{0, 1\}^n, & \text{if } S \text{ has } n \text{ elements,} \\ \cup_{j=1}^\infty \{0, 1\}^j, & \text{if } S \text{ is infinite,} \end{cases}$$

and

$$S^\pi := \{\underline{g}^e : e \in E\}.$$

The **preordering generated by S** in $\mathbb{R}[x]$ is defined by

$$T_S := \left\{ \sum_{s \in S^\pi} \sigma_s s : \sigma_s \in \sum \mathbb{R}[x]^2 \text{ for each } s \text{ and } \sigma_s \neq 0 \text{ for finitely many } s \right\}.$$

For $d \in \mathbb{N} \cup \{0\}$ we define the set

$$T_S^{(d)} := \left\{ \sum_{s \in S^\pi} \sigma_s s : \sigma_s \in \sum \mathbb{R}[x]^2 \text{ and } \deg(\sigma_s s) \leq d \text{ for each } s, \sigma_s \neq 0 \text{ for finitely many } s \right\}$$

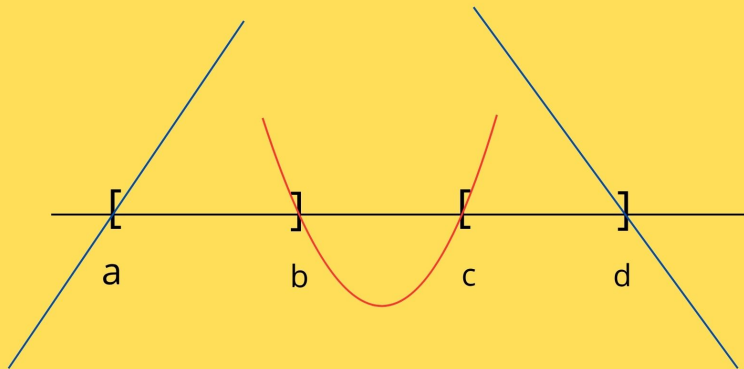
We call $T_S^{(d)}$ the **degree d truncation** of the preordering T_S .

The natural description

A set $S_K \subset \mathbb{R}[x]$ is the **natural description** of the closed set K , if it satisfies the following conditions :

- (a) If K has the least element $a \in \mathbb{R}$, then $x - a \in S_K$.
- (b) If K has the greatest element $b \in \mathbb{R}$, then $b - x \in S_K$.
- (c) For every $a, b \in K$, $a \neq b$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S_K$.
- (d) These are the only elements of S_K .

Natural description of $[a,b] \cup [c,d]$:



Nonsingular case

Theorem

Let $K \subseteq \mathbb{R}$ be a closed set and $\Lambda \subset \mathbb{R}$ be a finite or a countable closed set such that $\Lambda \cap \text{iso}(K) = \emptyset$. Let $L : \mathbb{R}[x]_{\leq 2k} \rightarrow \mathbb{R}$ be a linear functional and $L|_{T_{S_K}^{(2k)} \setminus \{0\}} > 0$, where S_K is the natural description of K . Then there exists an r -atomic measure $\mu \in \mathcal{M}_{L,K,\Lambda}^{(\text{fa})}$ with $k + 1 \leq r \leq 2k + 1$.

Moreover, assume that K is closed and semialgebraic¹ and write

$$\text{card}(\partial K) = 2\ell_1 + \ell_2, \ell_1 \in \mathbb{N} \cup \{0\}, \ell_2 \in \{0, 1\}.$$

Then r is at most :

- ① $k + 1$, if $K \in \{\mathbb{R}, [a, \infty), (-\infty, a]\}$ for some $a \in \mathbb{R}$.
- ② $k + \ell_1 + 1$, if K is bounded and has a non-empty interior.
- ③ $k + \ell_1 + \ell_2 + 1$, if K is bounded only from one side.
- ④ $k + \ell_1 + 2$, if K is unbounded from both sides.

1. A closed set $K \subseteq \mathbb{R}$ is **semialgebraic**, if it is of the form

$$K := \{x \in \mathbb{R} : p_1(x) \geq 0, \dots, p_m(x) \geq 0\}$$

for some $p_1, \dots, p_m \in \mathbb{R}[x]$.

Proof

Fact : $\text{Pos}_{\leq d}(K) = T_{S_K}^{(d)}$.

Since $\Lambda \cap \text{iso}(K) = \emptyset$, it follows that

$$\text{Pos}_{\leq 2k}(K) = \text{Pos}_{\leq 2k}(K \setminus \Lambda).$$

Note that

$$L|_{T_{S_K}^{(2k)} \setminus \{0\}} > 0 \Rightarrow L|_{\text{Pos}_{\leq 2k}(K) \setminus \{0\}} > 0.$$

Since Λ is closed, $K \setminus \Lambda$ is locally compact Hausdorff space.

By [Ph. di Dio, K. Sch.], L has a r -atomic K -rm μ such that $\mu(\Lambda) = 0$ and $r \in \{k+1, \dots, 2k+1\}$.

Singular case

Theorem : Let $K \subseteq \mathbb{R}$ be a closed set and $\Lambda \subset \mathbb{R}$ be a finite or a countable closed set such that $\Lambda \cap \text{iso}(K) = \emptyset$. Let $L : \mathbb{R}[x]_{\leq 2k} \rightarrow \mathbb{R}$ be a linear functional with $L|_{T_{S_K}^{(2k)} \setminus \{0\}} \not\equiv 0$, where S_K is the natural description of K .

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(A) $\mathcal{M}_{L,K,\Lambda} \neq \emptyset$.

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- (A) $\mathcal{M}_{L,K,\Lambda} \neq \emptyset$.
- (B) $\mathcal{M}_{L,K,\Lambda}^{(\text{fa})} \neq \emptyset$.
- (C) The following statements hold :
 - (i) $L|_{T_{S_K}^{(2k)} \setminus \{0\}} \geq 0$.

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(i) $L|_{T_{S_K}^{(2k)} \setminus \{0\}} \geq 0$.

(ii) If :

① $f_0 \in S^\pi$ is a polynomial of the lowest degree in S^π such that L_{f_0} is singular,

② $0 \neq p_{f_0}$ is a polynomial of the lowest degree such that $p_{f_0}^2 \in \ker L_{f_0}$,

then

$$\mathcal{Z}(f_0 p_{f_0}) \cap \Lambda = \emptyset.$$

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(iii) If K is unbounded, then

$$x^d p_{f_0}^2 \in \ker L_{f_0},$$

where f_0, p_{f_0} are as in (ii) and $d := 2k - \deg(f_0 p_{f_0}^2)$.

Moreover, if $\mathcal{M}_{L,K,\Lambda}^{(fa)} \neq \emptyset$, then the representing measure μ for L is unique and

$$\text{supp}(\mu) = \mathcal{Z}(f_0) \cup \mathcal{Z}(p_{f_0}).$$

2. Localize linear functional L_f is defined by $L_f(g) := L(fg)$

(C) \Rightarrow (B) :

1 : Let $K_1 = K \setminus \Lambda$. Since $\Lambda \cap \text{iso}(K) = \emptyset$, it follows that $\text{Pos}(K) = \text{Pos}(K_1)$.

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Let $\mu \in \mathcal{M}_{L_1, K}$. Since by assumption $0 = L_{f_0}(p_{f_0}^2)$, it follows by K -positivity of L that

$$0 = L_{f_0}(p_{f_0}) = (L_1)_{f_0}(p_{f_0}) = \int_K f_0 p_{f_0} d\mu,$$

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If $\text{supp}(\mu) \neq \mathcal{Z}(f_0) \cup \mathcal{Z}(p_{f_0})$, then either $\mathcal{Z}(f_0) \not\subseteq \text{supp}(\mu)$ or $\mathcal{Z}(p_{f_0}) \not\subseteq \text{supp}(\mu)$. In the first case there exists $f_1 \in S^\pi$ of lower degree than f_0 , such that L_{f_1} is also singular, which is a contradiction. In the second case there exists p of lower degree than p_{f_0} , such that $L_{f_0}(p^2) = 0$, which is a contradiction. Hence, $\text{supp}(\mu) = \mathcal{Z}(f_0) \cup \mathcal{Z}(p_{f_0})$ and $\mu \in \mathcal{M}_{L_1, K}$ is uniquely determined.

By $\mathcal{Z}(f_0 p_{f_0}) \cap \Lambda = \emptyset$, it follows that $\mu \in \mathcal{M}_{L_1, K, \Lambda}^{(\text{fa})}$. We separate 2 cases according to K and $\deg f_0$.

- If K is compact, then, by [Tchakaloff], L has a K -rm and hence μ must also represents $L(x^{2k})$.
- If K is unbounded, then $x^d p_{f_0}^2 \in \ker L_{f_0}$ ensures $L(x^{2k})$ is also a moment of μ , whence $\mu \in \mathcal{M}_{L, K, \Lambda}^{(\text{fa})}$.

This concludes the proof of the theorem.

Theorem (Curto, Fialkow)

Let $p(x, y) = x^2 + y^2 - 1$ and $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k}$, where $k \geq 2$. Then the following statements are equivalent :

- 1 β has a $\mathcal{Z}(p)$ -representing measure.
- 2 L_β is square-positive and the relations $\beta_{2+i,j} + \beta_{i,2+j} = \beta_{i,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 2$.

Idea of the proof : Take $x(t) = \frac{t^2-1}{t^2+1}$, $y(t) = \frac{2t}{t^2+1}$, $t \in \mathbb{R}$, a rational parametrization of $\mathcal{Z}(p)$, which is one-to-one and onto $\mathcal{Z}(p) \setminus \{(1, 0)\}$.

Reference

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Thank You For Your Attention !