Hausdorff moment net in two variables and toral m-isometric pairs

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  $(\beta_j)_0 = 1, (\beta_j)_1 = \beta_j$ and

$$(\beta_j)_{\alpha_j} = \beta_j(\beta_j - 1)\cdots(\beta_j - \alpha_j + 1), \quad \alpha_j \geqslant 2,\ j = 1, \ldots, n$$

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- $\binom{\beta}{\alpha} := \prod_{j=1}^n \binom{\beta_j}{\alpha_j},\ \alpha, \beta \in \mathbb{Z}_+^n$ with $\alpha \leqslant \beta$

Forward difference operator

For a net $\{a_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ and $j = 1, \ldots, n$, let $\triangle_j$ denote the *forward difference operator* given by

$$\triangle_j a_\alpha = a_{\alpha+\varepsilon_j} - a_\alpha, \quad \alpha \in \mathbb{Z}_+^n,$$

where ε_j stands for the n-tuple with jth entry equal to 1 and 0 elsewhere.

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where ε_j stands for the n-tuple with jth entry equal to 1 and 0 elsewhere. Note that $\triangle_1, \ldots, \triangle_n$ are mutually commuting. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, let $\triangle^\alpha$ denote the operator $\prod_{j=1}^n \triangle_j^{\alpha_j}$.

Joint completely monotone net

Definition

A net $\mathfrak{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ of nonnegative real numbers is said to be *joint completely monotone* if

$$(-1)^{|\beta|} \triangle^\beta a_\alpha \geqslant 0, \quad \alpha, \beta \in \mathbb{Z}_+^n.$$

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When $n = 1$, we simply refer to $\mathfrak{a}$ as a *completely monotone* sequence.We say $\mathfrak{a}$ is a *separate completely monotone* if for every $j \in \{1, \ldots, n\}$, $k \in \mathbb{Z}_+$,

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It is readily seen that a joint completely monotone net is separate completely monotone.

Example

For $t \in (0,1)$, consider the net

$$a_\alpha = t^{|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n.$$

Note that for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$,

$$\triangle_1 t^\alpha = t^{\alpha_2+\cdots+\alpha_n} \triangle_1 t^{\alpha_1} = t^{\alpha_2+\cdots+\alpha_n}(t^{\alpha_1+1} - t^{\alpha_1}) = t^{|\alpha|}(t-1) < 0.$$

More generally

$$(-1)^{|\beta|} \triangle^\beta a_\alpha = (-1)^{|\beta|} \triangle^\beta (t^{|\alpha|}) = t^{|\alpha|}(1-t)^{|\beta|} \geqslant 0, \quad \alpha, \beta \in \mathbb{Z}_+^n.$$

This shows that $\{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is a joint completely monotone net.

JCM $\Leftrightarrow$ HM

Theorem (Hildebrandt-Schoenberg, 1933)

A net $\mathfrak{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ of nonnegative real numbers is joint completely monotone if and only if it is a Hausdorff moment net, that is, if there exists a finite positive Borel measure μ concentrated on $[0,1]^n$ such that

$$a_\alpha = \int_{[0,1]^n} t^\alpha \mu(dt), \quad \alpha \in \mathbb{Z}_+^n. \tag{0.1}$$

If such a measure μ exists, then it is unique. We refer to μ as appearing in (0.1) as the representing measure of $\mathfrak{a}$.

The result in one variable is due to F. Hausdorff.

One variable results

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- For distinct non negative real numbers $a_0, \cdots, a_n$ and non-zero real numbers $b_0, \cdots, b_n$, consider the polynomial

$$p(x) = \prod_{k=0}^{n} (x + a_k + ib_k)(x + a_k - ib_k).$$

Then $\{1/p(l)\}_{l \in \mathbb{Z}_+}$ is never a Hausdorff moment sequence.

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Note that

$$\frac{1}{n^2+1} = \int_0^\infty e^{-nt} sin(t) dt = \int_0^1 t^n \frac{sin(-\ln t)}{t} dt, \quad n \in \mathbb{Z}_+.$$

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- Let $p(x) = (x+1)((x+a)^2 + b^2), a, b \in \mathbb{R}, b \neq 0$. Then the sequence $\{1/p(l)\}_{l \in \mathbb{Z}_+}$ is Hausdorff moment if and only if $a > 1$.

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Theorem (K. Ball, 1994)

Let $m \in \mathbb{N}$ *and*

$$f(x) = \frac{(x+a_1)(x+a_2)\dots(x+a_m)}{(x+b_1)(x+b_2)\dots(x+b_m)},$$

with $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m, 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_m$. *Then,* $\{f(n)\}_{n \in \mathbb{Z}_+}$ *is a completely monotone sequence provided*

$$\sum_{i=1}^{l} b_i \leqslant \sum_{i=1}^{l} a_i,$$

for every $l \in \{1, \dots, n\}$.

Question

Let n be a positive integer. For which polynomials $p, q : \mathbb{R}_+^n \to (0, \infty)$, the net $\{q(\alpha)/p(\alpha)\}_{\alpha \in \mathbb{Z}_+^n}$ is joint completely monotone? If $\{q(\alpha)/p(\alpha)\}_{\alpha \in \mathbb{Z}_+^n}$ is a Hausdorff moment net, then what is the representing measure of $\{q(\alpha)/p(\alpha)\}_{\alpha \in \mathbb{Z}_+^n}$?

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Example

Consider $p(x, y) = 1 + y + xy$. Since the reciprocal of any polynomial from $\mathbb{R}_+$ into $(0, \infty)$ of degree 1 is completely monotone, $\left\{\frac{1}{p(m,n)}\right\}_{m,n \in \mathbb{Z}_+}$ is separate completely monotone. Note that

$$\triangle_1 \triangle_2 \Big(\frac{1}{p}\Big)(m, n)\big|_{(m.n)=(0,0)} = -\frac{1}{6}.$$

This implies $\left\{\frac{1}{p(m,n)}\right\}_{m,n \in \mathbb{Z}_+}$ is not joint completely monotone. Thus, the joint complete monotonicity of $1/p$ may fail for a polynomial of bi-degree $(1, 1)$.

JCM in two variables

Theorem

For $a_j, b_j \in (0, \infty)$, $j = 0, \ldots, k$, let

$$a(x) = a_0 \prod_{j=1}^{k} (x + a_j), \;\; b(x) = b_0 \prod_{j=1}^{k} (x + b_j), \quad x \in \mathbb{R}_+.$$

Let $\{c(m)\}_{m \in \mathbb{Z}_+}$ be a sequence of positive real numbers. Assume that

(A1) $b_1 \leqslant a_1 \leqslant b_2 \leqslant a_2 \leqslant \ldots \leqslant b_k \leqslant a_k$,

(A2) $\left\{\frac{c(m)}{a(m)}\right\}_{m \in \mathbb{Z}_+}$ *is a completely monotone sequence.*

Then the net $\left\{\frac{c(m)}{b(m)+a(m)n}\right\}_{m,n \in \mathbb{Z}_+}$ is joint completely monotone.

Idea of the proof

- Note that for $m, n \in \mathbb{Z}_+$,

$$\frac{c(m)}{b(m)+a(m)n} = \int_{(0,1)} t^n t^{\frac{b(m)}{a(m)}-1} \frac{c(m)}{a(m)} dt.$$

- There exists non-positive real numbers $c_0, \cdots, c_n$ such that

$$\frac{b(m)}{a(m)} = c_0 + \sum_{j=1}^{k} \frac{c_j}{m+a_j}, \quad m \in \mathbb{Z}_+.$$

- For $t \in (0,1)$ and $m \in \mathbb{Z}_+$, $t^{\frac{b(m)}{a(m)}} = e^{\frac{b(m)}{a(m)} \ln t}$.
- For any real number $x > 0$,

$$\frac{1}{(n+x)^l} = \int_0^1 s^n \frac{s^{x-1}(-\log s)^{l-1}}{(l-1)!} ds, \quad l \geqslant 1, \; n \geqslant 0.$$

$c(m) = 1$

Theorem

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$$a(x) = a_0 \prod_{j=1}^{k} (x + a_j), \;\; b(x) = b_0 \prod_{j=1}^{k} (x + b_j), \quad x \in \mathbb{R}_+.$$

The following statements are valid:

(i) $\left\{ \frac{1}{b(m)+a(m)n} \right\}_{m,n \in \mathbb{Z}_+}$ *is a joint completely monotone net if*

$$b_1 \leqslant a_1 \leqslant b_2 \leqslant a_2 \leqslant \ldots \leqslant b_k \leqslant a_k,$$

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(ii) *if* $\left\{ \frac{1}{b(m)+a(m)n} \right\}_{m,n \in \mathbb{Z}_+}$ *is a joint completely monotone net, then*

$$\sum_{j=1}^{k} \frac{1}{a_j} \leqslant \sum_{j=1}^{k} \frac{1}{b_j}, \quad \prod_{j=1}^{k} b_j \leqslant \prod_{j=1}^{k} a_j, \quad \sum_{j=1}^{k} b_j \leqslant \sum_{j=1}^{k} a_j.$$

Bi-degree atmost $(1,1)$

Theorem

Let $p : \mathbb{R}^2_+ \to (0,\infty)$ *be a polynomial given by*

$$p(x,y) = a + bx + cy + dxy, \quad x, y \in \mathbb{R}^2_+,$$

where $a, b, c, d \in \mathbb{R}$. *Then* $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ *is a joint completely monotone net if and only if* $bc - ad \geqslant 0$. *Moreover, the representing measure of* $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ *is given by*

$$\begin{cases} \frac{s^{\frac{c}{d}-1}t^{\frac{b}{d}-1}}{d}\sum_{k=0}^{\infty}\left(\frac{bc-ad}{d^2}\right)^k \frac{(\log t \log s)^k}{(k!)^2}\, dsdt, & d \neq 0, \\ \frac{1}{b}s^{\frac{a}{b}-1} d\delta_{s^{c/b}}(t)\, ds, & b \neq 0,\ d = 0. \end{cases}$$

Theorem

For $a_j, b_j \in (0, \infty)$, $j = 0, 1, 2,$ *let*

$$a(x) = a_0(x + a_1)(x + a_2), b(x) = b_0(x + b_1)(x + b_2),$$

$x \in \mathbb{R}_+$ *with* $a_1 \leqslant a_2$ *and* $b_1 \leqslant b_2$. *Then* $\left\{\frac{1}{b(m)+a(m)n}\right\}_{m,n\in\mathbb{Z}_+}$ *is a joint completely monotone net provided*

$$(b_1 \leqslant a_1 \leqslant b_2 \text{ or } b_1 \leqslant a_2 \leqslant b_2) \text{ and } b_1 + b_2 \leqslant a_1 + a_2.$$

Bi-degree (2,1)

Subcase (b)

Theorem

For $a_0, a_1, b_0, b_1, b_2 \in (0, \infty)$, *let*

$$a(x) = a_0(x + a_1),\ b(x) = b_0(x + b_1)(x + b_2).$$

Then the net

$$\left\{ \frac{1}{b(m) + a(m)n} \right\}_{m,n \in \mathbb{Z}_+}$$

is joint completely monotone if and only if $b_1 \leqslant a_1 \leqslant b_2$.

Special case of bi-degree $(2, 2)$

Theorem

Let p be a polynomial given by $p(x, y) = a(x) + b(x)y + y^2$, where

$$a(x) = a_0(x + a_1)(x + a_2),\ b(x) = b_0(x + b_1),$$

$a_0, a_1, a_2, b_0, b_1 \in \mathbb{R}$ with $a_1 \leqslant a_2$. Assume that $p(m, n) > 0$ for every $m, n \in \mathbb{Z}_+$. Then $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint completely monotone net if and only if

$$4a_0 \leqslant b_0^2 \text{ and } a_0(a_2 - a_1)^2 \leqslant b_0^2(b_1 - a_1)(a_2 - b_1).$$

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- For a commuting n-tuple T on $\mathscr{H}$, define

$$T^* := (T_1^*, \ldots, T_n^*),\ T^\alpha := \prod_{j=1}^{n} T_j^{\alpha_j}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n.$$

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- A commuting n-tuple T is said to be a *toral expansion* (resp. a *toral contraction*) if $T_j^* T_j \geqslant I$ (resp. $T_j^* T_j \leqslant I$) for every $j \in \{1, \ldots, n\}$.

Toral m-isometry

Definition

Let $m \in \mathbb{Z}_+$. A commuting n-tuple $T = (T_1, \ldots, T_n)$ is a *toral m-isometry* if

$$\sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ 0 \leqslant \alpha \leqslant \beta}} (-1)^{|\alpha|} \binom{\beta}{\alpha} T^{*\alpha} T^{\alpha} = 0, \quad \beta \in \mathbb{Z}_+^n, \ |\beta| = m.$$

When $n = 1$, we simply refer T as an m-isometry. We say that a commuting n-tuple T is a *separate m-isometry* if $T_1, \ldots, T_n$ are m-isometries.

Case : $m, n = 2$

A commuting bounded 2-tuple $T = (T_1, T_2)$ is a toral 2-isometry if it satisfies the following equations:

$$I - 2T_1^* T_1 + T_1^{*^2} T_1^2 = 0, \tag{0.2}$$

$$I - T_1^* T_1 - T_2^* T_2 + T_2^* T_1^* T_2 T_1 = 0,$$

$$I - 2T_2^* T_2 + T_2^{*^2} T_2^2 = 0. \tag{0.3}$$

Note that by (0.2) and (0.3), T_1 and T_2 are 2-isometries.

Toral Cauchy dual of an operator

Let $T = (T_1, \ldots, T_n)$ is a commuting n-tuple consisting of left-invertible operators $T_1, \ldots, T_n$. We refer to the n-tuple $T^{\mathfrak{t}} := (T_1^{\mathfrak{t}}, \cdots, T_n^{\mathfrak{t}})$ as the *operator tuple torally Cauchy dual* to T, where

$$T_j^{\mathfrak{t}} := T_j(T_j^* T_j)^{-1}, \quad j = 1, \cdots, n.$$

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Fact

An operator $S \in B(\mathscr{H})$ is left-invertible if and only if S^*S is invertible in $B(\mathscr{H})$.

Joint subnormality of n-tuples

Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple on $\mathscr{H}$. We say that T is *jointly subnormal* if there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a commuting n-tuple N of normal operators $N_1, \ldots, N_n$ on $\mathscr{K}$ such that

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The Cauchy dual subnormality problem (for short CDSP) in n-variables asks whether the Cauchy dual of an m-isometric n-tuple is jointly subnormal.

Unilateral weighted n-shift

- Let $\mathbf{w} = \left\{ w_\alpha^{(j)} : j = 1, \ldots, n,\ \alpha \in \mathbb{Z}_+^n \right\}$ be a set of nonzero complex numbers.

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- Let $\mathscr{H}$ be a complex separable Hilbert space with orthonormal basis $\mathscr{E} = \{e_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$.
- A *weighted n-shift* $\mathscr{W} = (\mathscr{W}_1, \ldots, \mathscr{W}_n)$ with respect to $\mathscr{E}$ is defined by

$$\mathscr{W}_j e_\alpha := w_\alpha^{(j)} e_{\alpha+\varepsilon_j}, \quad j = 1, \ldots, n,$$

where ε_j is the m-tuple with 1 in the jth place and zeros elsewhere.

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  where ε_j is the m-tuple with 1 in the jth place and zeros elsewhere.
- $\mathscr{W}_1, \ldots, \mathscr{W}_n$ extend to densely defined linear operators on the linear span of $\{e_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$.

Unilateral weighted n-shift

- Let $\mathbf{w} = \left\{ w_\alpha^{(j)} : j = 1, \ldots, n,\ \alpha \in \mathbb{Z}_+^n \right\}$ be a set of nonzero complex numbers.
- Let $\mathscr{H}$ be a complex separable Hilbert space with orthonormal basis $\mathscr{E} = \{e_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$.
- A *weighted n-shift* $\mathscr{W} = (\mathscr{W}_1, \ldots, \mathscr{W}_n)$ with respect to $\mathscr{E}$ is defined by

$$\mathscr{W}_j e_\alpha := w_\alpha^{(j)} e_{\alpha+\varepsilon_j}, \quad j = 1, \ldots, n,$$

  where ε_j is the m-tuple with 1 in the jth place and zeros elsewhere.
- $\mathscr{W}_1, \ldots, \mathscr{W}_n$ extend to densely defined linear operators on the linear span of $\{e_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$.
- $\mathscr{W}_1, \ldots, \mathscr{W}_n$ extend boundedly to $\mathscr{H}$ if and only if

$$\sup_{\alpha \in \mathbb{Z}_+^n} |w_\alpha^{(j)}| < \infty, \quad j = 1, \ldots, n.$$

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- For $i, j = 1, \ldots, n$,

$$\mathscr{W}_i \mathscr{W}_j = \mathscr{W}_j \mathscr{W}_i \iff w_\alpha^{(i)} w_{\alpha+\varepsilon_i}^{(j)} = w_\alpha^{(j)} w_{\alpha+\varepsilon_j}^{(i)}, \quad \alpha \in \mathbb{Z}_+^n.$$

In what follows, we always assume the following

- $\mathscr{W}$ is a commuting n-tuple.
- The weight multi-sequence $\mathbf{w}$ of $\mathscr{W}$ consists of positive numbers.
- $\mathscr{W}$ extends boundedly to $\mathscr{H}$.

We indicate the weighted n-shift $\mathscr{W}$ with weight multi-sequence $\mathbf{w}$ by $\mathscr{W} : \{w_\alpha^{(j)}\}$.

Toral Cauchy dual of weighted n-shift

Let $\mathscr{W}: \{w_\alpha^{(j)}\}$ be a weighted n-shift such that $\mathscr{W}_j^*\mathscr{W}_j$ is invertible for every $j = 1, \ldots, n$. The operator tuple $\mathscr{W}^{\mathfrak{t}}$ torally Cauchy dual to the weighted n-shift $\mathscr{W}$ satisfies

$$\mathscr{W}_j^{\mathfrak{t}} e_\alpha = \frac{1}{w_\alpha^{(j)}}\, e_{\alpha+\varepsilon_j}, \quad j = 1, \ldots, n. \tag{0.4}$$

Moreover, by (0.4),

$$\|(\mathscr{W}^{\mathfrak{t}})^\alpha e_0\|^2 = \frac{1}{\|\mathscr{W}^\alpha e_0\|^2}, \quad \alpha \in \mathbb{Z}_+^n.$$

Proposition

Let $\mathscr{W}$ be a torally expansive weighted n-shift and let $\mathscr{W}^{\mathfrak{t}}$ be its toral Cauchy dual. Then $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $\left\{\frac{1}{\|\mathscr{W}^{\alpha} e_0\|^2}\right\}_{\alpha\in\mathbb{Z}_+^n}$ is a joint completely monotone net.

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Proposition

For a weighted 2-shift $\mathscr{W} : \{w_\alpha^{(j)}\}$, $\mathscr{W}$ is a toral m-isometry if and only if for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|\mathscr{W}^{\alpha}e_0\|^2 = \sum_{k=0}^{m-1} \sum_{\substack{\beta\in\mathbb{Z}_+^2 \\ |\beta|=k}} \frac{\triangle^{\beta}(\|\mathscr{W}^{\alpha}e_0\|^2)|_{\alpha=0}}{\beta!} (\alpha)_{\beta}.$$

Case $m = 3$

For $i, j \in \{0, 1, 2\}$, define

$$\rho_{ij} = \triangle_1^i \triangle_2^j (\|\mathscr{W}^\alpha e_0\|^2)|_{\alpha=0}.$$

Fix a_1, a_2, b_1, b_2 and c_1 as follows

$$a_1 = \rho_{10} - \frac{\rho_{20}}{2}, \quad a_2 = \frac{\rho_{20}}{2}, \quad b_1 = \rho_{11}, \quad b_2 = \rho_{01} - \frac{\rho_{02}}{2}, \quad c_1 = \frac{\rho_{02}}{2}. \qquad (0.5)$$

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Proposition

For a weighted 2-shift $\mathscr{W} : \{w_\alpha^{(j)}\}$, $\mathscr{W}$ is a toral 3-isometry if and only if for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|\mathscr{W}^\alpha e_0\|^2 = 1 + a_1\alpha_1 + a_2\,\alpha_1^2 + (b_1\alpha_1 + b_2)\alpha_2 + c_1\,\alpha_2^2 \tag{0.6}$$

If (0.6) holds, then the following holds

1. $\mathscr{W}_2$ is a 2-isometry if and only if for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|\mathscr{W}^{\alpha} e_0\|^2 = 1 + a\,\alpha_1 + b\,\alpha_1^2 + (c + d\,\alpha_1)\,\alpha_2,$$

   where a, b, c and d are given by

$$a = \rho_{10} - \frac{\rho_{20}}{2}, \quad b = \frac{\rho_{20}}{2}, \quad c = \rho_{01}, \quad d = \rho_{11}.$$

2. $\mathscr{W}_1$ and $\mathscr{W}_2$ are 2-isometry if and only if for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|\mathscr{W}^{\alpha} e_0\|^2 = 1 + a\,\alpha_1 + (b + c\,\alpha_1)\,\alpha_2,$$

   where a, b, c and d are given by

$$a = \rho_{10}, \quad b = \rho_{01}, \quad c = \rho_{11}.$$

CDS of toral 3-isometric weighted 2-shift

Theorem

Let $\mathscr{W} : \{w_\alpha^{(j)}\}$ be a torally expansive toral 3-isometric weighted 2-shift and let $\mathscr{W}^{\mathfrak{t}}$ be the operator tuple torally Cauchy dual to $\mathscr{W}$. The following statements holds:

(a) *Assume that $\mathscr{W}$ is a separate 2-isometry. The operator tuple $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if*

$$b_1 \leqslant a_1 b_2.$$

(b) *Assume that $\mathscr{W}_1$ is not a 2-isometry. The operator tuple $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $a_1 > 0, a_1^2 \geqslant 4a_2$,*

$$(2a_2 b_2 - a_1 b_1)^2 \leqslant (b_1^2 - 4a_2 c_1)(a_1^2 - 4a_2),$$

and any one of the following holds:

(i) $b_1 = 0, b_2 = 0, c_1 = 0,$
(ii) $b_1 > 0, b_2 > 0.$

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Thank You